MATH2007B - Test 3 - 14:35 -15:25, Nov 4, Tuesday

Name: Student Number:

Total: 15 marks
Closed book, Non-programmable calculators are allowed!

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1. (4 points) Multiple choices, no partial marks.

   (i) Converting the polar point $(r, \theta) = (4, \frac{\pi}{3})$ to Cartesian coordinates (x, y), then $x =$

   (a) π (b) 1 (c) 2 (d) -1 (e) $2\sqrt{3}$.

   Solution: (c). $x = r\cos\theta = 4\cos\dfrac{\pi}{3} = 2.$

   (ii) Converting the Cartesian point $(-2, -2)$ to polar point (r, θ), then $\theta =$

   (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$ (e) $\frac{2\pi}{3}$.

   Solution: (c). Note that the point is in the third quadrant.

$$\tan\theta = \frac{y}{x} = 1. \Rightarrow \theta = \frac{5\pi}{4}.$$

   (iii) Given the polar curve $r = 4\cos\theta$. Which of the following is a Cartesian point on the curve?

   (a) $(-2, 2)$ (b) $(0, 4)$ (c) $(-2, 4)$ (d) $(-1, 2)$ (e) $(2, 2)$

   Solution: (e)
   $x = r\cos\theta = 4\cos^2\theta, y = r\sin\theta = 4\cos\theta\sin\theta \Rightarrow$

$$y^2 = 16\cos^2\theta\sin^2\theta = 4x(\sin^2\theta) = 4x(1-\cos^2\theta) = 4x(1-\frac{x}{4}) = 4x - x^2.$$

   (iv) Which of the following gives the length of the curve $r = 2\theta$, $0 \le \theta \le 2\pi$?

   (a) $\int_0^{2\pi} \sqrt{1+\theta^2}d\theta$ (b) $2\int_0^{2\pi} \sqrt{4+\theta^2}d\theta$ (c) $\int_0^{2\pi} \sqrt{1+4\theta^2}d\theta$ (d) $2\int_0^{2\pi} \sqrt{1+\theta^2}d\theta$

   Solution: (d).

$$L = \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2}d\theta = \int_0^{2\pi} \sqrt{4+4\theta^2}d\theta = 2\int_0^{2\pi} \sqrt{1+\theta^2}d\theta.$$

2. (5 points=2+2+1) A curve C is defined by the parametric equations $x = t^4 + 1, y = t^4 + 16t$. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, and determine the values of t for which the curve C is concave up.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3+16}{4t^3} = 1 + \frac{4}{t^3}.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-12t^{-4}}{4t^3} = -3t^{-7} = -\frac{3}{t^7}.$$

Note that $\frac{d^2y}{dx^2} > 0$ for all $t < 0$, the curve C is concave up for all $t < 0$.

3. (3 points) Given the polar curve C: $r = 2 + \cos\theta$, where $0 \le \theta \le \pi/2$. Find the area of the region enclosed by the curve C and the rays: $\theta = 0$ and $\theta = \pi/2$.

Solution: We have

$$\begin{aligned} A &= (1/2)\int_0^{\pi/2} r^2 d\theta \qquad\qquad (1\ point) \\ &= (1/2)\int_0^{\pi/2} (2+\cos\theta)^2 d\theta = (1/2)\int_0^{\pi/2} (4 + 4\cos\theta + \cos^2\theta) d\theta \\ &= \pi + \int_0^{\pi/2} 2\cos\theta d\theta + (1/2)\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta \\ &= \pi + 2 + \pi/8 = 2 + \frac{9\pi}{8}. \end{aligned}$$

4. (3 points) Find the equation of the tangent line to the polar curve $r = 4\sin\theta$ at $\theta = \frac{\pi}{6}$.

Solution:

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}.$$

At $\theta = \frac{\pi}{6}$,

$$\frac{dy}{dx} = \sqrt{3}, \qquad\qquad (1\ point)$$

$$x = r\cos\theta = 4\sin\theta\cos\theta = \sqrt{3}, y = r\sin\theta = 4\sin\theta\sin\theta = 1. \qquad\qquad (1\ point)$$

Let

$$y = \sqrt{3}x + b, \quad 1 = \sqrt{3}(\sqrt{3}) + b, \Rightarrow b = -2, \Rightarrow$$
$$y = \sqrt{3}x - 2.$$