1. Compute the derivative of the following functions.
YOU DO NOT NEED TO SIMPLIFY YOUR ANSWERS.

[3] a) $f(x) = \sqrt{x}\arctan(x)$.

Solution:

$$f'(x) = \frac{\arctan(x)}{2\sqrt{x}} + \frac{\sqrt{x}}{1+x^2}.$$

[3] b) $f(x) = \dfrac{2^x - 1}{\ln(x)}$.

Solution:

$$f'(x) = \frac{\ln(2)2^x \ln(x) - \frac{2^x-1}{x}}{\ln^2(x)}.$$

[4] c) $f(x) = \cos\left(x^{-1} - 5x^2 + 3\right)$.

Solution:

$$f'(x) = -\sin\left(x^{-1} - 5x^2 + 3\right)\left(-x^{-2} - 10x\right).$$

[4] d) $f(x) = e^{\sin(3x^4-1)}$.

Solution:

$$f'(x) = e^{\sin(3x^4-1)}\cos(3x^4 - 1)12x^3.$$

[3] 2. For $f(x) = \dfrac{2}{x}$, use the **limit definition** to compute the derivative $f'(x)$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h \to 0} \frac{\frac{-2h}{x^2+xh}}{h} = \lim_{h \to 0} \frac{-2}{x^2 + xh} = \frac{-2}{x^2}.$$

[3] 3. Given the function $f(x) = \tan(x)$, find the linear approximation of $f(x)$ at $a = 0$ and then estimate $f(0.15)$.

Solution: We have $f(0) = 0$. Since $f'(x) = \sec^2(x)$, we have $f'(0) = 1$. Thus

$$L_0(x) = f'(a)(x - a) + f(a) = x.$$

Now,

$$f(0.15) \approx L_0(0.15) = 0.15.$$

[5] 4. A curve on the plane is given by $x^2 + xy + y^2 - 5x = 4$. Find the slope of the tangent line on the curve at the point $(0, 2)$.

Solution: Taking d/dx we get

$$2x + y + xy' + 2yy' - 5 = 0.$$

Solving for y' e get

$$y' = \frac{5 - 2x - y}{x + 2y}.$$

Thus, the slope at $(0, 2)$ is

$$\left(\frac{5 - 2x - y}{x + 2y}\right)\Bigg|_{(0,2)} = \frac{3}{4}.$$

[3] 5. A cylindrical tank with radius 5 m is being filled with water at a rate of $3\,\mathrm{m}^3/\mathrm{min}$. How fast is the height of the water increasing?

Solution: The volume of the cylinder is

$$V = \pi r^2 h = 25\pi h.$$

Since $v' = 5$ and

$$V' = 25\pi h'$$

we conclude

$$h' = \frac{V'}{25\pi} = \frac{3}{25\pi}.$$

[5] 6. A 24 m^2 rectangular garden is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimension for the outer rectangle will require the smallest total length of fence?

Solution: Let x be the side of the outer rectangle parallel to the splitting fence. Let y be the other side of the outer rectangle. Then

$$24 = xy.$$

Then

$$y = \frac{24}{x}.$$

We want to maximize

$$P = 3x + 2y = 3x + \frac{48}{x}.$$

Taking $P'(x)$ we get

$$P'(x) = 3 - \frac{48}{x^2}.$$

The critical points are $x = \pm 4$.

Since

$$P'(x) = \frac{96}{x^3},$$

we have that $P''(4) > 0$, so we get a minimum at $x = 4$ and $y = 6$.

7. Compute the following limits.

[4] a) $\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}}$.

Solution:

$$\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$$

[5] b) $\lim_{x\to 0^+} \sin(x)\ln(x)$.

Solution:

$$\lim_{x\to 0^+} \sin(x)\ln(x) \overset{\infty\cdot 0}{=} \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sin(x)}} \overset{\infty/\infty}{=} \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} \overset{0/0}{=} \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos(x) - x\sin(x)} = 0.$$

[5] 8. Find the domain, intercepts and asymptotes for $f(x) = \dfrac{2}{e^x + 1}$.

Solution: Domain: $\mathbb{R}$.

x-intercepts: DNE.

y-intercepts: $y = 2$.

VA: DNE.

HA: $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 2$.

[5] 9. Find the intervals of increase and decrease and the local maximum and minimum for the function $f(x)$ whose **first derivative** is given by $f'(x) = \dfrac{(x-5)^2(x+1)}{x-1}$.

Solution: Intervals determined by $x = \{-1, 1, 5\}$. Thus

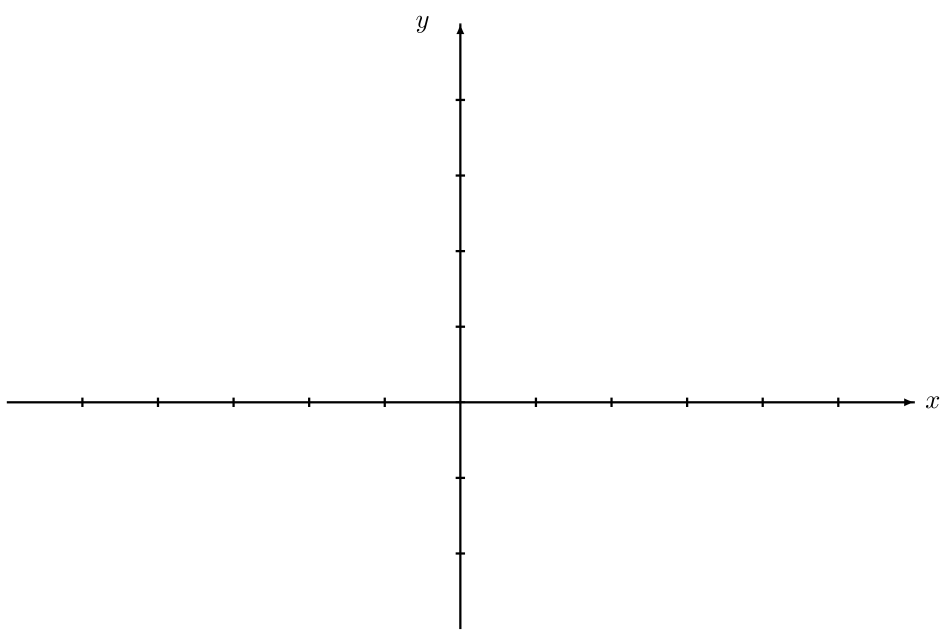
Interval	f'	f
$(-\infty, -1)$	Positive	Increasing
$(-1, 1)$	Negative	Decreasing
$(1, 5)$	Positive	Increasing
$(5, \infty)$	Positive	Increasing

[**4**] 10. Sketch a graph of the function $y = f(x)$ with the following properties.

- Domain : $(-\infty, 1) \cup (1, \infty)$.
- y-intercept: $y = 2$.
- x-intercepts: $x = -1$.
- $\lim_{x \to 1} f(x) = \infty$.
- $\lim_{x \to -\infty} f(x) = -\infty, \qquad \lim_{x \to \infty} f(x) = 2.$
-

Interval	f'
$(-\infty, 1)$	Positive
$(1, \infty)$	Negative

Interval	f''
$(-\infty, -1)$	Negative
$(-1, 1)$	Positive
$(1, \infty)$	Positive



[3] 11. Find $f(x)$ if $f'(x) = 8x + \sin(x)$ and $f(0) = \pi$.

Solution:

$$f(x) = 4x^2 - \cos(x) + C.$$

Since

$$\pi = -1 + C$$

we conclude

$$f(x) = 4x^2 - \cos(x) + \pi + 1.$$

[3] 12. Estimate the area under the graph of $f(x) = \sqrt{x}$ from 0 to 4 using four approximating rectangles and right endpoints.

Solution: Since $\Delta_x = 1$, we have

$$Area = f(1) + f(2) + f(3) + f(4) = 1 + \sqrt{2} + \sqrt{3} + 2 = 3 + \sqrt{2} + \sqrt{3}.$$

[3] 13. If $f(x)$ and $g(x)$ are continuous functions such that

$$\int_2^0 f(x)\,dx = \pi, \quad \int_0^2 g(x)\,dx = 6\pi,$$

find the value of

$$\int_0^2 (3g(x) + f(x))\,dx.$$

Solution:

$$\int_0^2 (3g(x) + f(x))\,dx = 3(6\pi) - \pi = 17\pi.$$

[3] 14. Find $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \sqrt{1+t^3}\,dt$.

Solution:

$$\frac{d}{dx}\int_0^{x^2} \sqrt{1+t^3}\,dt = 2x\sqrt{1+(x^2)^3}.$$

15. Find the following indefinite integrals using any appropriate integration technique.

[3] a) $\displaystyle\int \frac{4x^2 + x\sqrt{x}}{x^3}\,dx.$

Solution:

$$\int \frac{4x^2 + x\sqrt{x}}{x^3}\,dx = \int \left(\frac{4}{x} + x^{-3/2}\right)\,dx = 4\ln|x| - \frac{2}{\sqrt{x}} + C.$$

[4] b) $\displaystyle\int xe^{x^2+1}\,dx.$

Solution: Take

$$u = x^2 + 1,$$

so

$$dx = 2x\,dx.$$

Then

$$\int xe^{x^2+1}\,dx = \frac{1}{2}\int e^u\,du = \frac{e^u}{2} + C = \frac{e^{x^2+1}}{2} + C.$$

[4] c) $\int x^2 \ln x \, dx$.

Solution: Take

$$u = \ln(x), \qquad v = x^2 \, dx,$$

so

$$du = 1/x \, dx, \qquad v = x^3/3.$$

Then

$$\int x^2 \ln x \, dx = \frac{x^3 \ln(x)}{3} - \int \frac{x^2}{3} \, dx = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + C.$$

[6] d) $\displaystyle\int \frac{9}{x(x^2+9)}\,dx$.

Solution: We have

$$\frac{9}{x(x^2+9)} = \frac{A_1}{x} + \frac{A_2x+B_2}{x^2+9} = \frac{A_1(x^2+9)+x(A_2x+B_2)}{x(x^2+9)}$$

Then we must have

$$9 = A_1(x^2+9) + x(A_2x+B_2).$$

For $x=0$ we have $9 = 9A_1$, so

$$A_1 = 1.$$

Then

$$9 = (x^2+9) + x(A_2x+B_2) = x^2(A_2+1) + xB_2 + 9.$$

Then

$$A_2 = -1, \qquad B_2 = 0.$$

Hence

$$\int \frac{9}{x(x^2+9)}\,dx = \int \left(\frac{1}{x} - \frac{x}{x^2+9}\right)dx = \ln|x| - \frac{1}{2}\ln|x^2+9| + C.$$

[6] e) $\displaystyle\int \frac{x^3}{\sqrt{16-x^2}}\,dx.$

Solution: Take

$$x = 4\sin(\theta), \qquad dx = 4\cos(\theta)\,d\theta.$$

We have

$$\sqrt{16-x^2} = 4\cos(\theta), \qquad x^3 = 4^3\sin^3(\theta).$$

Thus

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{4^3\sin^3(\theta)}{4\cos(\theta)}\,4\cos(\theta)\,d\theta = \int 4^3\sin^3(\theta)\,d\theta = \int 4^3\sin^2(\theta)\sin(\theta)\,d\theta = \int 4^3\left(1-\cos^2(\theta)\right)\,\sin(\theta)\,d\theta$$

Take

$$u = \cos(\theta) \qquad du = -\sin(\theta)\,d\theta.$$

Then

$$\begin{aligned}\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int 4^3\left(1-\cos^2(\theta)\right)\,\sin(\theta)\,d\theta = -\int 4^3\left(1-u^2\right)\,du = -4^3\left(u-\frac{u^3}{3}\right)+C \\ = -4^3\left(\cos(\theta)-\frac{\cos^3(\theta)}{3}\right)+C \\ = -4^3\left(\frac{(16-x^2)^{1/2}}{4}-\frac{(16-x^2)^{3/2}}{4^3\cdot 3}\right)+C.\end{aligned}$$

[**4**] 16. Evaluate $\displaystyle\int_0^1 \frac{e^x+1}{e^x+x}\,dx$.

Solution: Take

$$u = e^x + x,$$

so

$$du = (e^x + 1)\,dx.$$

Then

$$\int_0^1 \frac{e^x+1}{e^x+x}\,dx = \int_1^{e+1} \frac{1}{u}\,du = \ln|u|\Big|_1^{e+1} = \ln(e+1).$$

[5] 17. Find the following limit

$$\lim_{x\to 0} \frac{\int_0^x (e^t - 1 - t)\, dt}{x^3}.$$

Solution:

$$\lim_{x\to 0} \frac{\int_0^x (e^t - 1 - t)\, dt}{x^3} \overset{0/0}{=} \lim_{x\to 0} \frac{e^x - 1 - x}{3x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{e^x - 1}{6x} \overset{0/0}{=} \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}.$$

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