

MATH2007B - Test 2 - 14:35 -15:25, Oct. 14, Tuesday

Name:

Student Number:

Total: 15 marks

Closed book, Non-programmable calculators are allowed!

1. (4 points) Multiple choices, no partial marks.

(i) $\int_1^{+\infty} 3x^2 e^{-x^3} =$

- (a) $\frac{1}{e}$ (b) 1 (c) -1 (d) e (e) Divergent.

Solution: (a). $\int_1^{+\infty} 3x^2 e^{-x^3} = -e^{-x^3} \Big|_1^{+\infty} = e^{-1}$.

(ii) Find all p values such that $\int_3^{+\infty} \frac{1}{x^p}$ is convergent.

- (a) $p > 1$ (b) $p < 1$ (c) $p \geq 1$ (d) $p \leq 1$ (e) $p \geq 0$.

Solution: (a).

$$\int_3^{+\infty} \frac{1}{x^p} = \frac{x^{-p+1}}{-p+1} \Big|_3^{+\infty} = \begin{cases} \frac{3^{-p+1}}{p-1}, & \text{if } p > 1; \\ \text{divergent}, & \text{if } p \leq 1. \end{cases}$$

(iii) $\int_0^1 \frac{1}{\sqrt{x}} dx =$

- (a) 0 (b) 1 (c) 2 (d) 1/2 (e) Divergent.

Solution: (c). $\int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_0^1 = 2$.

(iv) Which of the following is a correct substitution to solve $\int_0^1 \sqrt{x^2 + 4} dx$?

- (a) $x = 2 \sin t$ (b) $x = 2 \sec t$ (c) $x = 4 \sin t$ (d) $x = 4 \sec t$ (e) $x = 2 \tan t$

Solution: (e). Let $x = 2 \tan t$. Then $\sqrt{x^2 + 4} = \sqrt{4 \sec^2 t} = 2 \sec t$.

2. (2 points) Evaluate $\int \frac{2x - 3}{(x + 1)(x - 4)} dx$.

Solution: By Partial Fraction Method,

$$\frac{2x - 3}{(x + 1)(x - 4)} = \frac{1}{x + 1} + \frac{1}{x - 4}.$$

Thus

$$\int \frac{2x-3}{(x+1)(x-4)} dx = \int \frac{1}{x+1} + \int \frac{1}{x-4} dx = \ln|x+1| + \ln|x-4| + C.$$

3. (3 points) Evaluate $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$.

Solution: Let $x = 2 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then when $x = 0$, $\theta = 0$; when $x = 2$, $\theta = \frac{\pi}{2}$.
 $\sqrt{4-x^2} = 2 \cos \theta$, $dx = 2 \cos \theta d\theta$. Hence

$$\begin{aligned} \int_0^2 \frac{1}{\sqrt{4-x^2}} dx &= \int_0^{\pi/2} \frac{1}{2 \cos \theta} (2 \cos \theta d\theta) \\ &= \int_0^{\pi/2} d\theta = \frac{\pi}{2}. \end{aligned}$$

4. (3 points) Evaluate $\int \frac{4x^2 + 5x - 8}{(x+1)^2(x-2)} dx$.

Solution: Let

$$\frac{4x^2 + 5x - 8}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}, \Rightarrow A = 2, B = 3, C = 2.$$

Hence

$$\begin{aligned} \int \frac{4x^2 + 5x - 8}{(x+1)^2(x-2)} dx &= \int \left(\frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{2}{x-2} \right) dx \\ &= 2 \ln|x+1| - \frac{3}{x+1} + 2 \ln|x-2| + C. \end{aligned}$$

5. (3 points) Determine if the integral $\int_5^\infty x e^{-x} dx$ is convergent or divergent and evaluate if it is convergent.

Solution: $I = \lim_{t \rightarrow \infty} \int_5^t x e^{-x} dx$ (integration by parts: let $u = x$ and $dv = e^{-x} dx$)

$$\begin{aligned} &= \lim_{t \rightarrow \infty} \left(x(-e^{-x}) \Big|_5^t - \int_5^t -e^{-x} dx \right) \\ &= \lim_{t \rightarrow \infty} (-xe^{-x} - e^{-x}) \Big|_5^t \\ &= \lim_{t \rightarrow \infty} -(x+1)e^{-x} \Big|_5^t \\ &= \lim_{t \rightarrow \infty} (-(t+1)e^{-t} + 6e^{-5}) \\ &= 6e^{-5} \text{ (where } \lim_{t \rightarrow \infty} (t+1)e^{-t} = 0 \text{ by L'Hospital's Rule).} \end{aligned}$$