MAT1341 Notes - By Eric Hua

Contents

4.1-4.3. Vectors, Lines and Planes

The Vector Space (Euclidean Space) $\mathbb{R}^n$:

$$\mathbb{R}^n = \{(u_1, u_2, ..., u_n) | u_i \in \mathbb{R}\} = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} | u_i \in \mathbb{R} \right\}.$$

- Distance $d(\vec{u}, \vec{v}) = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$.
- Length (norm, magnitude) $||(u_1, u_2, ..., u_n)|| = \sqrt{u_1^2 + \cdots + u_n^2}$.
- Sum: Let $\vec{u} = (u_1, ..., u_n)$, $\vec{v} = (v_1, ..., v_n)$, then $\vec{u} + \vec{v} = (u_1 + v_1, ..., u_n + v_n)$.
- Scalar multiple: Let c be a scalar, then $c\vec{u} = (cu_1, cu_2, ..., cu_n)$.
- Dot product (inner product): Let $\vec{u} = (u_1, ..., u_n)$, $\vec{v} = (v_1, ..., v_n)$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$.
- unit vector: $||\vec{u}|| = 1$.
- Projection: The projection of $\vec{u}$ onto $\vec{v}$ is

$$proj_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \right) \vec{v}, \quad comp_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||}.$$

- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0 \leq \theta \leq \pi$, then $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||}$.
- Orthogonal: $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$.

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}, \quad \vec{u} + (-\vec{u}) = \vec{0}$
- $(cd)\vec{u} = c(d\vec{u})$
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

- $1\vec{u} = \vec{u}, \quad (-1)\vec{u} = -\vec{u}, \quad 0\vec{u} = \vec{0}$
- $\vec{u} // \vec{v} \Leftrightarrow \vec{v} = c\vec{u}$
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{w} \cdot (\vec{u} + \vec{v}) = \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v}$
- $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
- $\vec{u} \cdot \vec{0} = 0$
- $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$:

Example 1. *Let $\vec{u} = (1, 2, -2)$. Find the unit vector which has the same direction as $\vec{u}$.*

- Direction angles to the three axis and direction cosines of vectors.:

$$\cos\alpha = \frac{\vec{u} \cdot \vec{i}}{||\vec{u}||\,||\vec{i}||}, \quad \cos\beta = \frac{\vec{u} \cdot \vec{j}}{||\vec{u}||\,||\vec{j}||}, \quad \cos\gamma = \frac{\vec{u} \cdot \vec{k}}{||\vec{u}||\,||\vec{k}||}.$$

  They satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \frac{\vec{u}}{||\vec{u}||} = (\cos\alpha, \cos\beta, \cos\gamma).$$

Example 2. *Let $\vec{u} = (1, 2, -2)$, $\vec{v} = (-2, -2, 1)$. Write $\vec{u} = \vec{u_1} + \vec{u_2}$ s.t. $\vec{u_1} \perp \vec{v}$, $\vec{u_2} // \vec{v}$.*

Example 3. *Let $\vec{u} = (1, 2, -2)$, $\vec{v} = (-2, -2, 1)$, Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.*

Solution:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||} = \frac{-8}{9}.$$

The Cross Product

- Cross product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, +u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

- Orthogonal: $\vec{u} \times \vec{v} \perp \vec{u}$, $\vec{u} \times \vec{v} \perp \vec{v}$.

Example 4. *Find a vector that is orthogonal to both* $\vec{u} = (1, 2, -1)$*,* $\vec{v} = (0, 2, 3)$*.*

Solution: Any scalar multiple of $\vec{u} \times \vec{v} = (8, -3, 2)$.

Properties: Let c be a scalar.

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.
- $||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{u}|| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$
- $||\vec{u} \times \vec{v}||$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.
- $||\vec{u} \cdot (\vec{v} \times \vec{w})||$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$. If it is 0, then these three vectors are in the same plane.

Example 5. *Find the area of the parallelogram determined by* $\vec{u} = (1, 2, -1)$*,* $\vec{v} = (0, 2, 3)$*.*

Solution: $A = ||\vec{u} \times \vec{v}|| = ||(8, -3, 2)|| = \sqrt{77}$.

Example 6. *Find the area of the triangle with vertices* $P(1, 2, 3)$*,* $Q(-3, 2, 1)$*, and* $R(2, 4, 5)$*.*

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.

$$A = \frac{1}{2}||\vec{PQ} \times \vec{PR}|| = ||(4, 6, -8)|| = \sqrt{29}.$$

Example 7. *Find the volume of the parallelepiped spanned by* $\vec{u} = (1, 2, 3), \vec{v} = (1, 3, 2), \vec{w} = (1, 2, 2)$*.*

Solution: Volume $= ||\vec{u}\cdot(\vec{v}\times\vec{w})|| = 1$.

Line:

A line is determined by a point and a vector (direction vector) parallel to the line. Let $P(p_1, p_2, p_3)$ be a point on the line L. Let $\vec{v}$ be a nonzero vector which is parallel L.

- Vector equation (Point-parallel form): $\vec{r} = \vec{p} + t\vec{v}$, $t \in \mathbb{R}$, $\vec{p} = (p_1, p_2, p_3)$, $\vec{r} = (x, y, z)$.
- Parametric equations: $x = p_1 + tv_1$, $y = p_2 + tv_2$, $z = p_3 + tv_3$.
- Symmetric form: $\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$.

Remark. (Two-point form). If a line goes through two points P and Q, then $\vec{r}(t) = \vec{p} + t(\vec{q} - \vec{p})$, where $\vec{q}$, $\vec{p}$ are the position vectors of Q, P.

Example 8. *Find the equation of the line through $P(1, 2, 3)$ and $Q(3, 1, 1)$.*

Two lines L_1 and L_2 can be

- parallel
- intersected
- skewed

Example 9. *Show that the intersection between $L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t$ and $L_2 : x = 2s, y = 3 + s, z = -3 + 3s$ is $(16/5, 23/5, 9/5)$.*

Example 10. *Show that the two lines $L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t; L_2 : x = 2s, y = 3 + s, z = -3 + 4s$ are skew lines.*

Solution. (1) The two direction vectors are $v_1 = (1, 3, -1)$ and $v_2 = (2, 1, 4)$. They are not parallel.

(2) No intersection: From "$x = x$" and "$y = y$" we have $1 + t = 2s, -2 + 3t = 3 + s \Rightarrow s = 1.6, t = 2.2$. By L_1, $z = 1.8$; by L_2, $z = 3.4$.

Plane:

A plane Π is determined by a point (x_0, y_0, z_0) and a normal vector $\vec{n} \neq \vec{0}$ which is perpendicular to the plane.

- Vector equation: $\vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}$, where
  - $\vec{r} = (x, y, z)$ is a position vector for any point (x, y, z) on the plane,
  - $\vec{r_0} = (x_0, y_0, z_0)$ is the position vector for the given point (x_0, y_0, z_0) on the plane,
  - $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ are two non-parallel direction vectors parallel to the plane,
  - s, t are scalars in $\mathbb{R}$.
- Parametric equations: $x = x_0 + ta_1 + sb_1$, $y = y_0 + ta_2 + sb_2$, $z = z_0 + ta_3 + sb_3$.
- Scalar equation (standard equation): $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C) = \vec{a} \times \vec{b}$. This equation comes from the point-normal form:

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

Example 11. *Find the equation of the plane through three points $P(1, 2, 3)$, $Q(-3, 2, 1)$, and $R(2, 4, 5)$.*

Solution: $\vec{PQ} = (-4, 0, -2)$, $\vec{PR} = (1, 2, 2)$. $\vec{n} = \vec{PQ} \times \vec{PR} = (4, 6, -8)$. Thus

Scalar equation (standard equation): $4(x-1)+6(y-2)-8(z-3) = 0, \Rightarrow 2x+3y-4z = -4$.

Vector equation: $\vec{v} = (1, 2, 3) + t(-4, 0, -2) + s(1, 2, 2), \quad t, s \in \mathbb{R}$.

Parametric equations: $x = 1 - 4t + s$, $y = 2 + 2s$, $z = 3 - 2t + 2s, \quad t, s \in \mathbb{R}$.

Example 12. ***Special planes:*** *xy plane, yz plane, xz plane; planes parallel to z-axis; planes perpendicular to z-axis; plane containing two parallel lines; vector equation of a plane from scalar equation.*

Angle between two planes:

Two planes are parallel if their normal vectors are parallel. The angle between two planes is defined as the angle between their normal vectors:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}.$$

Example 13. *Find the angle between two planes $x - 3y - 2z = 3$ and $3x + 5y + 8z = 1$.*

Solution: $\cos\theta = \frac{-2}{\sqrt{7}}, \Rightarrow \theta = 180^o - 41^o = 139^o$.

Intersection between two planes:

If the two planes with normal vectors $\vec{n_1}$ and $\vec{n_2}$ are not parallel, then the intersection is a line with direction vector $\vec{n_1} \times \vec{n_2}$.

Example 14. *Find the intersection between two planes $x-3y-2z = 2$ and $2x+y+3z = 1$.*

Solution: (1) $\vec{n_1} \times \vec{n_2} =< -7, 7, 7 >= 7 < -1, 1, 1 >$.

(2) To find one intersection point, we let $z = 0$. Then $x = 5/7, y = -3/7$. So the parametric equation of the line is:

$$(x, y, z) = (5/7, -3/7, 0) + t(-1, 1, 1), \quad t \in \mathbb{R}.$$

Line and Plane:

Example 15. *Find the intersection between the line $L : x = 1 + t, y = -2 + 3t, z = 4 - t$ and the plane $3x + 5y + 8z = 1$.*

Example 16. *Find the parametric equations of the line containing the point $(1, 2, 3)$ and which is parallel to the two planes $x - 3y - 2z = 3$ and $3x + 5y + 8z = 1$.*

Distance between a point and a plane:

A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane. Let $P(p_1, p_2, p_3)$ be a point, let Π be: $ax + by + cz = d$. Then the distance between them is:

$$d = \frac{|ap_1 + bp_2 + cp_3 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 17. *Show that the distance between $P(1, 2, 3)$ and the plane $2x - 2y - z = 1$ is 2.*

Remark. Finding the distance between two parallel planes: This equals the distance from any point in one plane to the other plane.

Distance between a point P **and a line** L**:** Pick up a point Q from L. Let $\vec{v}$ be a direction vector of L. Then the distance is

$$d = ||\vec{QP} - proj_{\vec{v}}\vec{QP}||.$$

Example 18. *Find the distance between* $P(3, -2, 5)$ *and the line* $L : (x, y, z) = (0, 0, 1) + t(1, 1, 1)$, $t \in \mathbb{R}$.

Solution: Let $Q = (0, 0, 1)$. $\vec{QP} = (3, 2, 4)$. $\vec{v} = (1, 1, 1)$. $proj_{\vec{v}}\vec{QP} = (5/3, 5/3, 5/3)$. Thus the distance is

$$d = ||\vec{QP} - proj_{\vec{v}}\vec{QP}|| = ||(4/3, -11/3, 7/3)|| = \frac{\sqrt{186}}{3}.$$

Special angles:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Appendix A: Complex Numbers

Complex number is

$$z = a + bi, \quad a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1.$$

Addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The conjugate of $z = a + bi$ is: $\bar{z} = a - bi$. We have

$$z\bar{z} = a^2 + b^2.$$

The modulus of z is defined by

$$|z| = r = \sqrt{a^2 + b^2}.$$

Polar form: $z = a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta)$, where $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$.

Properties

1. $\bar{z} = z$ if and only if z is real,
2. $\overline{z + w} = \bar{z} + \bar{w}$,
3. $\overline{zw} = \bar{z}\bar{w}$,
4. $|wz| = |w||z|$,
5. $|z + w| \leq |z| + |w|$ (triangle inequality),
6. let $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$ (De Moivre's Theorem).
7. let $z = r(\cos\theta + i\sin\theta)$, then $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$, where $k = 0, ..., n-1$.

Example 19. *Simplify* $\frac{3+4i}{4-5i}$.

Solution: $\frac{3+4i}{4-5i} = \frac{(3+4i)(4+5i)}{(4-5i)(4+5i)} = \frac{-8+31i}{41} = -\frac{8}{41} + \frac{31}{41}i$.

Example 20. *Find the polar form:* $\frac{\sqrt{2}-\sqrt{2}i}{-2+2\sqrt{3}i}$.

Solution: The numerator $\sqrt{2}-\sqrt{2}i$: $r=2$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\sin\theta = \frac{-\sqrt{2}}{2}$. $\Rightarrow \theta = \frac{7\pi}{4}$. Thus $\sqrt{2}-\sqrt{2}i = 2e^{\frac{7\pi}{4}i}$.

Similarly, the denominator $-2+2\sqrt{3}i = 4e^{\frac{2\pi}{3}i}$.

$$\frac{\sqrt{2}-\sqrt{2}i}{-2+2\sqrt{3}i} = \frac{2}{4}e^{\frac{7\pi}{4}i-\frac{2\pi}{3}i} = \frac{1}{2}e^{\frac{13\pi}{12}i}.$$

Example 21. *Let $z=\sqrt{3}+i$. Calculate z^6.*

Example 22. *Solve the following equation $2x^2-4x+4=0$.*

Example 23. *Construct a polynomial with real coefficients such that $3-2i$ is a root.*

6.1 Vector Spaces

Vector Spaces.

Definition. Let V be a set of elements. If the operations addition $\oplus$ and scalar multiplication $\otimes$ are defined in V satisfying the following axioms:

A1. If $\vec{u}, \vec{v} \in V$, then $\vec{u} \oplus \vec{v} \in V$.

A2. If $\vec{u}, \vec{v} \in V$, then $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.

A3. If $\vec{u}, \vec{v}, \vec{w} \in V$, then $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.

A4. There exists an element, denoted by $\vec{0}$, in V, such that $\vec{0} \oplus \vec{u} = \vec{u}$ for every $\vec{u}$.

A5. For every $\vec{u} \in V$, there exists an element, denoted by $-\vec{u}$, in V such that $\vec{u} \oplus (-\vec{u}) = \vec{0}$.

S1. Operation scalar multiplication is defined for every number c and every $\vec{u}$ in V, and $c \bigodot \vec{u} \in V$.

S2. Operation scalar multiplication satisfies the distributive law: $c \bigodot (\vec{u} \oplus \vec{v}) = c \bigodot \vec{u} \oplus c \bigodot \vec{v}$.

S3. Operation scalar multiplication satisfies the second distributive law: $(c + d) \bigodot \vec{u} = c \bigodot \vec{u} \oplus d \bigodot \vec{u}$.

S4. Operation scalar multiplication satisfies the associative law: $(cd) \bigodot \vec{u} = c \bigodot (d \bigodot \vec{u})$.

S5. For every element $u \in V$, $1 \bigodot \vec{u} = \vec{u}$.

Then V is called a vector space.

Example 24. *1. $\mathbb{R}^n$ with the usual operations is a vector space.*

2. $\mathbb{C}^n$ with the usual operations is a vector space.

3. P=the set of all polynomials with the usual operations are vector space.

4. M_{mn} = the set of all m by n matrices with the usual operations is a vector space.

5. $F(\mathbb{R})$ of all real-valued functions on $\mathbb{R}$ with the usual operations is a vector space.

6. $F[a, b]$ of all real-valued functions on $[a, b]$ with the usual operations is a vector space.

Example 25. *1. $\mathbb{Z}$ of integers with the usual operations is NOT a vector space. For example, $\sqrt{2}(3)$ is not integer.*

2. $\mathbb{R}^2$ with $c \bigodot (x, y) = (cx, 0)$ is not a vector space. For example, $1 \bigodot (2, 3) = (2, 0) \neq (2, 3)$.

Proposition 1. *(i)* $\vec{w} + \vec{v} = \vec{u} + \vec{v}$ *implies* $\vec{w} = \vec{u}$.

(ii) $0\vec{v} = \vec{0}$.

(iii) $c\vec{0} = \vec{0}$.

(iv) $a\vec{v} = \vec{0}$ *implies* $a = 0$ *or* $\vec{v} = \vec{0}$.

Example 26. *On* $\mathbb{R}^2$ *we define addition and scalar multiplication as*

$$\begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x+u-2 \\ y+v \end{bmatrix}, \quad k \bigodot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx-k+1 \\ ky \end{bmatrix}.$$

Then the zero vector is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$*, the negative of* $\begin{bmatrix} x \\ y \end{bmatrix}$ *is* $\begin{bmatrix} -x+4 \\ -y \end{bmatrix}$.

Example 27. $\mathbb{R}^3$ *with the scalar multiplication* $k \bigodot (x, y, z) = (kx, yz, ky)$ *is not a vector space.*

Solution: It does not satisfies Axioms S4 and S5.

6.2+5.1 Subspaces and Spanning Sets

Subspaces

A set U is a subspace of a vector space V if U is a vector space with respect to the operations of V.

Theorem 1. *U is a subspace of V if*

(i) the zero vector is in U;

(ii) if x is in U, then ax is in U for any scalar a, and

(iii) if x, y are in U, then x + y is in U.

Example 28.

- $S = \{(x, y, z) \mid x + z = y, x, y, z \in \mathbb{R}\}$.

  Solution:

  - *Notice that* $(0, 0, 0) \in S$ *since* $0 + 0 = 0$.
  - *If* (a, b, c) *and* (a', b', c') *are in* S*, by definition we have*

$$a + c = b \qquad \text{and} \qquad a' + c' = b' \tag{1}$$

Now, since $(a,b,c)+(a',b',c')=(a+a',b+b',c+c')$, *from the identities (1) we obtain that*

$$(a+a')+(c+c')=(a+c)+(a'+c')=b+b',$$

so S *is closed under addition.*

– *If* (a,b,c) *is in* S *and* α *is any scalar, we have that* $\alpha(a,b,c)=(\alpha a,\alpha b,\alpha c)$, *so from identity (1) we conclude that* $\alpha a+\alpha c=\alpha(a+c)=\alpha b$.

This implies that S *is closed under scalar multiplication.*

Thus, S *is a subspace of* $\mathbb{R}^3$.

- $H=\left\{\begin{bmatrix} a \\ a+1 \\ a-1 \end{bmatrix} \middle| a\in\mathbb{R}\right\}$ *is not a subspace of* $\mathbb{R}^3$;

Proof. *(1)* $\vec{0}\notin H$. *In fact, if* $\begin{bmatrix} a \\ a+1 \\ a-1 \end{bmatrix}=0$, *then* $a=0$, $a-1=0$, $a+1=0$, *a contradiction.*

(2) H *is not closed under addition: for example,* $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\in H$, *but*

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}+\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}=\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}\neq\begin{bmatrix} 3 \\ 3+1 \\ 3-1 \end{bmatrix}.$$

(3) H *is not closed under scalar multiplication: for example,* $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\in H$, *but*

$$3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}=\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}\neq\begin{bmatrix} 3 \\ 3+1 \\ 3-1 \end{bmatrix}.$$

- $H=\left\{\begin{bmatrix} a-1 \\ b \\ c \end{bmatrix} \middle| a,b,c\in\mathbb{R}\right\}$ *is a subspace of* $\mathbb{R}^3$.

Example 29. • $\{\vec{0}\}$ *is a subspace. A subspace that is not* $\{\vec{0}\}$ *is a proper subspace.*

- *A line through the origin in the space is a subspace; A plane through the origin in the space is a subspace.*
- $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a = 3b + 2c, a, b, c \in R \right\}$ *is a subspace of* M_{22}.
- P_n*, the set of all polynomials with degree* $\leq n$*, is a subspace of* P.
- $S = \{p \in P_2 | p(1) = 0\}$ *is a subspace of* P.
- $S = \{p \in P_2 | p(1) = 3\}$ *is not a subspace of* P.
- $C(\mathbb{R})$ *of all real-valued continuous functions on* $\mathbb{R}$ *with the usual operations is a subspace of* $F(\mathbb{R})$.

Example 30. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. *Then the set of matrices in* M_{22} *that are commutative with* A *is a subspace of* M_{22}.

Solution: Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be commutable with A. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = s \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Linear Combinations and Spanning Sets

Given a set of vectors $\vec{v}_1, \cdots, \vec{v}_m$ in the vector space V,

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m | \quad c_1, c_2, ..., c_m \in \mathbb{R}\}$$

is a subspace.

Proposition 2. *(i) If* $X \in S$*, then* $X \in spanS$.
(ii) If a subspace W contains every vector in S, then W contains span S.
As an example of using the second property, $span\{X+Y, X, Y\} = span\{X, Y\}$.
(iii) If $\vec{b}$ *is a linear combination of* $v_1, v_2, ..., v_k$*, then*
$spanS = \{\vec{b}, v_1, v_2, , v_k\} = spanS = \{v_1, v_2, ..., v_k\}$.
(iv) $R^n = span\{E_1, E_2, ..., E_n\}$.

Example 31. *(i) Verify that [1 2 0 1]*^{T} *is in span{[2 1 2 0]*^{T} *, [0 -3 2 2]*^{T} *}.*
(ii) Verify that the set of vectors S = {[1 2 3]^T*, [-1 0 1]*^{T} *, [2 1 -1]*^{T}*} spans* R^3.
Solution: For any [a b c]^T *in* R^3*, The corresponding system is consistent.*
(iii) Find a,b such that X = [a b a+b a-b]^T *is in span{A, B}, where A = [1 0 1 1]*^{T}*, B = [0 1 1 -1]*^{T}.

(iv) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$*. Then the set of matrices in* M_{22} *that are commutative with* A *is a subspace of* M_{22}*, which is spanned by* $\left\{ \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Example 32. $P_2 = span\,\{2, 3, x, 3x, 1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$.

6.3+5.2 Linear Independence and Dimension

Definition 1. *A set of vectors* $\{\vec{v_1}, \cdots, \vec{v_m}\}$ *in a vector space* V *is linearly independent if the vector equation*

$$x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_m\vec{v_m} = \vec{0}$$

implies that $x_1, x_2, \cdots, x_m = 0$. *The set is said to be linearly dependent if there is a non-trivial solution to the vector equation.*

Example 33. *Given* $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. *Show that* $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ *is linearly dependent and find the linear combination.*

Solution: $-2\vec{v_1} + \vec{v_2} - \vec{v_3} = \vec{0}$.

Example 34.

1. *In* $C(\mathbb{R})$, *the set* $\left\{\sin^2 x, \cos^2 x, 2\right\}$ *is linearly dependent.*
2. *In* P_2, *the set* $\{1, x, x^2\}$ *is linearly independent.*
3. *In* P_2, *the set* $\{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$ *is linearly dependent.*

Example 35. *Let* $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ *be linearly independent. Determine if the following set is linearly independent or dependent:* $S = \{\vec{v_1} - \vec{v_2}, \vec{v_2} - \vec{v_3}, \vec{v_3} - \vec{v_4}, \vec{v_4} - \vec{v_1}\}$.

Solution: We need to set up the equation

$$c_1(\vec{v_1} - \vec{v_2}) + c_2(\vec{v_2} - \vec{v_3}) + c_3(\vec{v_3} - \vec{v_4}) + c_4(\vec{v_4} - \vec{v_1}) = \vec{0}, \Rightarrow$$

$$(c_1 - c_4)\vec{v_1} + (-c_1 + c_2)\vec{v_2} + (-c_2 + c_3)\vec{v_3} + (-c_3 + c_4)\vec{v_4} = \vec{0}, \Rightarrow$$

$$c_1 = c_2 = c_3 = c_4. \quad e.g., c_1 = c_2 = c_3 = c_4 = 1.$$

Thus dependent.

Theorem 2.

1. *A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.*
2. *A set of two or more vectors is linearly dependent if and only if at least one vector may be written as a linearly combination of the others.*
3. *If a set contains more vectors than entries in each vector, then the set is linearly dependent.*

4. *If the zero vector is in a set of vectors, then the set of vectors is linearly dependent.*

5. *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^n$ *is linearly independent if and only if the row-reduced echelon form of the matrix* $[\vec{v}_1 \ \cdots \ \vec{v}_m]$ *contains only distinct elementary columns.*

6. *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^m$ *is linearly independent if and only if* $\det[\vec{v}_1 \ \cdots \ \vec{v}_m] \neq 0$.

Fundamental Theorem: Let U be a subspace of a vector space V. If U is spanned by m vectors and U contains k linearly independent vectors, then $k \leq m$.

Invariant Theorem: Let U be a subspace of $\mathbb{R}^n$. If $\{X_1, ..., X_k\}$ and $\{Y_1, ..., Y_m\}$ are two bases of U, then $k = m$.

Definition 2. *A basis for a subspace V is a linearly independent set of vectors that spans V. The number of vectors in a basis for a subspace V is called the dimension of V and is denoted by* $\dim V$.

Example 36. *Find a basis to each of the following subspaces:*

$$\begin{aligned} U &= \{(a+2b+3c, a+c, b+a+2c, a-b) | a, b, c \in \mathbb{R}\} \\ V &= \{(a, b, c, d) | a+2b = c, 3b-2c = d; a, b, c, d \in \mathbb{R}\} \end{aligned}$$

Solution:

- $U = \{a(1,1,1,1) + b(2,0,1,-1) + c(3,1,2,0) | a, b, c \in \mathbb{R}\}$
  $= Span\{(1,1,1,1), (2,0,1,-1), (3,-1,0,0)\}$.

  Next we show that the set of three vectors $S = \{(1,1,1,1), (2,0,1,-1), (3,-1,0,0)\}$ is linearly dependent. If

  $$a(1,1,1,1) + b(2,0,1,-1) + c(3,-1,0,0) = \vec{0},$$

  then

  $$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

  Thus a basis is $\{(1,1,1,1), (2,0,1,-1)\}$, vectors containing leading 1's.

- For the subspace V, from $a+2b=c, 3b-2c=d$ we imply that $d=-2a-b$. Thus $V=\{(a,b,a+2b,-2a-b)|a,b\in\mathbb{R}\}=\{a(1,0,1,-2)+b(0,1,2,-1)|a,b\in\mathbb{R}\}=Span\{(1,0,1,-2),(0,1,2,-1)\}$.

  Similarly we can show that the set of two vectors $T=\{(1,0,1,-2),(0,1,2,-1)\}$ is linearly independent. Thus T is a basis of V.

Example 37.

1. *The set $\{1,x,x^2,...,x^n\}$ is the standard basis of P_n.*
2. *Let E_{ij} be the matrix of $m\times n$ where the (i,j) entry is 1, all other entries are 0. Then the set $\{E_{11},...,E_{mn}\}$ is the standard basis of M_{mn}.*

Example 38. *Find a basis to the following subspace of M_{22}:*

$$W = \left\{\left[\begin{array}{cc} 2x & 3x \\ y+z & y+z-x \end{array}\right]\middle| x,y,z\in\mathbb{R}\right\}$$

Solution:

$$\begin{aligned} W &= \left\{x\left[\begin{array}{cc} 2 & 3 \\ 0 & -1 \end{array}\right]+y\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right]+z\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right]|x,y,z\in\mathbb{R}\right\} \\ &= Span\left\{\left[\begin{array}{cc} 2 & 3 \\ 0 & -1 \end{array}\right],\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right],\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right]\right\}=Span\left\{\left[\begin{array}{cc} 2 & 3 \\ 0 & -1 \end{array}\right],\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right]\right\}. \end{aligned}$$

Next we show that the set of two vectors $S=\left\{\left[\begin{array}{cc} 2 & 3 \\ 0 & -1 \end{array}\right],\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right]\right\}$ is linearly independent. Set up the equation

$$a\left[\begin{array}{cc} 2 & 3 \\ 0 & -1 \end{array}\right]+b\left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right]=\vec{0}.$$

then

$$\left[\begin{array}{cc} 2a & 3a \\ b & b-a \end{array}\right]=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right],$$

$$\Rightarrow 2a=0, 3a=0, b=0, b-a=0,$$

$$\Rightarrow a=0, b=0.$$

Thus S is independent, which is a basis of W.

Example 39. *In P_2, the set $B = \{1 + x, 1 - x + 3x^2, 1 + 3x - x^2\}$ is linearly independent. Express $p(x) = 2 - 5x - x^2$ as a linear combination of the vectors in B.*

Definition 3. *The dimension of a vector space V is defined to be the number of vectors in a basis. We write it as $\dim V$.*

Example 40.

1. $\dim\{\vec{0}\} = 0$.
2. $\dim P_n = n + 1$.
3. $\dim M_{mn} = mn$.

Example 41. *Extend the following linearly independent set $\{1 + x + x^2, 1 + 3x - x^2\}$ to a basis of P_2.*

Theorem 3. *Let V be a subspace space with $\dim V = k$. Then*

1. *Any set of more than k vectors in V is linearly dependent.*
2. *Any set of less than k vectors in V can not span V.*
3. *Every basis for V has exactly k vectors.*

1.1 Solutions and Elementary Operations

Definition 4. *A* ***linear equation*** *in variables* $x_1, x_2, \ldots, x_n$ *has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ *are the equation's coefficients and* $d \in \mathbb{R}$ *is the constant. An* n*-tuple* $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ *is a* ***solution*** *of, or satisfies, that equation if substituting the numbers* s_1, ..., s_n *for the variables gives a true statement:* $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.

A ***system of linear equations (or linear system)***

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ *if that* n*-tuple is a solution of all of the equations in the system.*
A linear system is called inconsistent if it has no solution. Otherwise it is called consistent.

Finding the set of all solutions is solving the system.

Example 42. *David inherited \$50,000 and invested part of it in a money market account which pays 6% annually, and part in a mutual fund which pays 5% annually. After one year, he received a total of \$2,700 in simple interest from the two investments. Find the amount David invested in each category.*

Solution: Let

$x =$ The amount of money invested in the money market account. $y =$ The amount of money invested in a mutual fund.

Then

$$\begin{array}{rcl} x + y & = & 50000 \\ 0.06x + 0.05y & = & 2700 \end{array}$$

The solution is $x = 20,000$, $y = 30,000$.

Definition 5. *If we have two linear systems and they have the same solution set then the two linear systems are called* ***equivalent****.*

Elementary operations: There are three types of elementary operations to a system of linear equations.

1. Replacement: Replace an equation by the sum of itself and the multiple of another equation.

2. Interchange: Interchange two equations.

3. Scaling: Multiply an equation by a non-zero constant.

Theorem 4. *Elementary operations will result in an equivalent system.*

Matrices:

Definition 6. *An $m \times n$ (m by n)* matrix *A with m rows and n columns with entries in $\mathbb{R}$ is a rectangular array of the form*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $\forall (i,j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}, \quad a_{ij} \in \mathbb{R}$.

As a shortcut, we often use the notation $A = [a_{ij}]$ to denote the matrix A with entries a_{ij}. Notice that when we refer to the matrix we put parentheses—as in "$[a_{ij}]$," and when we refer to a specific entry we do not use the surrounding parentheses—as in "a_{ij}."

Example 43. $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ *is a 2×3 matrix and* $B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$ *is a 3×2 matrix.*

- Diagonal matrix: Except for entries on diagonal (main diagonal), all other entries are 0,

- Zero matrix: all entries are 0,

- Identity matrix I_n: all entries on diagonal are 1, other entries are 0.

Coefficient Matrix and Augmented Matrix:

To solving linear systems, we put all the coefficients of each variable aligned in columns to get the **coefficient matrix**. By adding an additional column to the coefficient matrix consisting of the values on the right hand side of the equal sign to give the **augmented matrix**.

Elementary row operations: There are three types of elementary row operations.

1. Replacement: Replace one row by the sum of itself and the multiple of another: $R_i \to R_i + cR_j$.
2. Interchange: Interchange two rows: $R_i \leftrightarrow R_j$.
3. Scaling: Multiply all entries in a row by a non zero constant: $R_i \to cR_i$.

Definition 7. *Two matrices are row equivalent if one matrix can be transformed into another matrix by a sequence of elementary row operations.*

1.2 Gaussian Elimination

A **leading entry of a row**: is the leftmost, nonzero entry in the row (nonzero row).

Row echelon form: A rectangular matrix is in row-reduced echelon form if it satisfies the following three properties.

1. All non zero rows are above any zero rows.
2. The leading entry in each nonzero row is 1, which is called leading 1.
3. Each leading 1 is in a column to the right of the leading 1 above.

Reduced row echelon form: A rectangular matrix is in reduced row echelon form if it satisfies the 4th condition:

4. Each leading 1 is the only nonzero entry in its column.

We are interested in performing row operations until one of these two matrix structures arises.

Uniqueness of the Reduced Echelon Form: Each matrix is row equivalent to one and only one reduced echelon matrix.

Properties:

- The augmented matrix of a consistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} \cdots & * & * \end{bmatrix}.$$
- The augmented matrix of a inconsistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}.$$

Remark: The variables correspond to leading 1's, are called **leading variables (basic variables)**.

Gaussian elimination:

1. Carry the augmented matrix to the reduced row-echelon form;

2. The system is inconsistent if a row is [0 0 0 ... 1];

3. Otherwise, assign parameters to non-leading variables, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables (the unknowns) in terms of parameters.

Example 44. *Solve the following system by using elementary row operations (row reduction):*

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & -2 \\ & & y & + & 3z & = & 7 \\ x & & & - & 5z & = & -9 \end{array}$$

We can start by going to echelon form:

$$\text{augmented matrix } = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -5 & -9 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & -3 & -7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -5 & -9 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We write this back to a new linear system:

$$\begin{cases} x & = & -9 + 5z \\ y & = & 7 - 3z \\ z & = & free \end{cases}$$

Set $z = s$ (parameter), then the general solution of the system is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 + 5s \\ 7 - 3s \\ s \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

Rank:

Definition 8. *The rank of a matrix A = the number of leading entries in echelon form = the number of leading 1s in the row-reduced echelon form = the number of nonzero rows in the row-reduced echelon form.*

$$\text{For example, } rank \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 4 & 7 & 4 \\ 0 & 0 & -8 & -14 & -8 \end{bmatrix} = 2.$$

Theorem 5. *Any linear system has,*

1. *no solution*
2. *one solution*
3. *infinitely many solutions.*

In case 1, the linear system is called inconsistent. In cases 2 and 3, the linear system is called consistent.

Theorem 6. *For a linear system $A\vec{x} = \vec{b}$, where A is $m \times n$, let p be the rank of the coefficient matrix, let q be the rank of the augmented matrix. Then the system has*

1. *no solution, if $p < q$;*
2. *only one solution if $p = q = n$;*
3. *infinitely many solutions if $p = q < n$.*

Example 45. *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= c - 1 \end{aligned}$$

(i) Find value(s) of k and c such that the system has no solution.
(ii) Find value(s) of k and c such that the system has only one solution.
(iii) For the value(s) of k and c such that the system has infinitely many solutions.

Solution: (i)

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & k & c-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & k-9 & c-1 \end{bmatrix}$$

(i) If $k = 9$, $c \neq 1$, then no solution.
(ii) For $k \neq 9$, the system has only one solution.
(iii) When $k = 9$, $c = 1$, infinitely many solutions.

1.3. Homogeneous Equations

Definition 9. *A linear system $A\vec{x} = \vec{b}$ is called homogeneous if $\vec{b} = \vec{0}$. Otherwise, it is non-homogeneous. Let $\vec{x}$ be a solution of $A\vec{x} = \vec{0}$. If $\vec{x} = \vec{0}$, then $\vec{x}$ is called a trivial solution; if $\vec{x} \neq \vec{0}$, then $\vec{x}$ is called a non-trivial solution.*

Basic Solutions and linear combination

Whenever a solution set is described explicitly with vectors, we say that the solution is in parametric vector form.

Example 46. *Consider the following system of linear equations*

$$\begin{aligned} x_1 - 2x_2 - 9x_3 + 5x_4 &= 0 \\ x_2 + 2x_3 - 6x_4 &= 0. \end{aligned}$$

Describe the solution set in parametric vector form.

Solution:

$$\text{augmented matrix} = \begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}.$$

Hence the leading variables (basic variables) are x_1 and x_2. The non-leading variables x_3 and x_4 are parameters: $x_3 = s$ and $x_4 = t$. Then

$$\begin{aligned} x_1 &= 5s + 7t \\ x_2 &= -2s + 6t. \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5s + 7t \\ -2s + 6t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5s \\ -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 7t \\ 6t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$

Here $\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$ are called basic solutions.

Remark. Any nonzero multiple of a basic solution is also a basic solution. But we treat them as same basic solution.

Given a set of vectors $\vec{v}_1, \cdots, \vec{v}_m$. For $c_1, c_2, ..., c_m \in \mathbb{R}$,

$$\{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

is called a **linear combination** of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$.

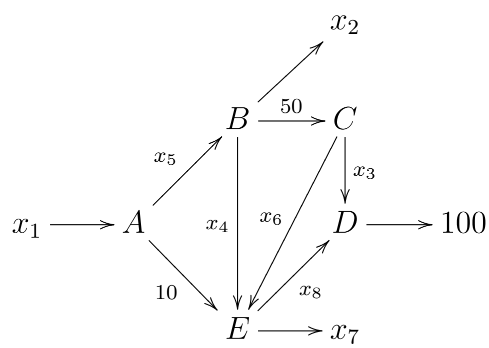
Proposition 3. *Consider a homogeneous system with n variables, m equations. Let r be the number of the leading variables.*

- *If $n > m$, then there are infinitely many solutions.*
- *The system has $n - r$ basic solutions ($n - r$ free variables). Every solution of the system is a linear combination of these basic solutions.*

1.4 An Application to Network Flow

Junction Rule: At each junction, total flow in = total flow out.

Example 47. *Consider the traffic flow described by the following diagram. The letters A through E label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.*



(a) Construct a linear system describing the traffic flow , including all constraints on the variables $x_i, i = 1, \ldots, 8$.

Solution: Set "flow in" = "flow out" at each intersection:

$$\begin{array}{lrl} A & x_1 & = x_5 + 10 \\ B & x_5 & = x_2 + x_4 + 50 \\ C & 50 & = x_3 + x_6 \\ D & x_3 + x_8 & = 100 \\ E & 10 + x_4 + x_6 & = x_7 + x_8 \end{array}$$

Constraints: x_i $(i = 1, ..., 8)$ are non-negative integers.

Remark. In some other textbooks, they add a row for "total flow in=total flow out". Actually that is not necessary.

(b) Carry the augmented matrix to the reduced row echelon form, and find the general flow pattern (i.e., the general solution of the linear system with more constraints).

Solution:
$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -90 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \end{array}\right]$$

$$\begin{cases} x_1 = x_5 + 10 \\ x_2 = x_5 - x_7 - 90 \\ x_3 = -x_8 + 100 \\ x_4 = x_7 + 40 \\ x_5 = \text{free} \\ x_6 = x_8 - 50 \\ x_7 = \text{free} \\ x_8 = \text{free} \end{cases}$$

i.e.,

$$\begin{cases} x_1 = s + 10 \\ x_2 = s - t - 90 \\ x_3 = -u + 100 \\ x_4 = t + 40 \\ x_5 = s \\ x_6 = u - 50 \\ x_7 = t \\ x_8 = u \end{cases},$$

where $s, t, u \in \mathbb{R}$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 10 \\ -90 \\ 100 \\ 40 \\ 0 \\ -50 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad s, t, u \in \mathbb{R}.$$

Constraints: $s - t \geq 90$, $50 \leq u \leq 100$.
Remark: For a consistent system, if it has free variables, it has infinite solutions; if no free variable, then only one solution.

(c) What is the minimum and maximum number of traffic along the road ED?

Solution: Since $x_6 = x_8 - 50 \geq 0$, $x_8 \geq 50$. From $x_3 = -x_8 + 100 \geq 0$ we imply that $x_8 \leq 100$. Thus $50 \leq x_8 \leq 100$.

(d) Suppose that due to road work, the flow along ED is limited to a maximum of 70 cars per minute. What is the maximum possible flow along CE?

Solution: Since $x_6 = x_8 - 50$, from $x_8 \leq 70$ it follows that $x_6 \leq 20$, i.e., the maximum number of cars along CE is 20 cars per minutes.

2.1-2.4 Matrix Algebra

2.1 Matrices

Scalar multiplication and addition of matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, r be a number. Then

$$rA = [ra_{ij}], \quad A + B = [a_{ij} + b_{ij}].$$

Remark. You can only add matrices of the same size. Also, two matrices are equal if they are the same size and corresponding entries are equal.

Example 48. $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 \\ 8 & 10 & 12 \end{bmatrix}$. $3 \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & 6 \\ 0 & 9 \end{bmatrix}$.

Properties:

Let A, B, C be matrices of the same size and let r and s be scalars.

1. A+B = B+A
2. (A+B)+C = A+(B+C)
3. A+0 = A
4. r(A+B) = rA + rB
5. (r+s)A = rA + sA
6. r(sA) = (rs)A

Transpose of a matrix: Given a matrix A then transpose of A is a matrix denoted by A^T, whose rows are the columns of A and whose columns are the rows of A.

Example 49. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Properties of Transpose:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ where r is a scalar.
4. If $A = A^T$, then A is called symmetric. If A and B are symmetric, then $A + B$ is symmetric.

2.2 Equations, Matrices

Matrix Vector multiplication:

Let $A = [A_1\ A_2\ \cdots A_n]$, $X = [x_1\ x_2\ \cdots x_n]^T$. Then we define

$$AX = x_1 A_1 + \cdots + x_n A_n.$$

Proposition 4. $A(X+Y) = AX + AY; \quad A(cX) = cAX; \quad (A+B)X = AX + BX.$

Matrix equation of a linear system

$AX = B$ is called matrix equation. $AX = 0$ is called the associated homogeneous system.

Example 50. *The linear system:*

$$\begin{aligned} x_1 - x_4 &= 1 \\ x_1 - 2x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ x_2 + 5x_3 &= 4 \end{aligned}$$

$$\Leftrightarrow$$

The matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Theorem 7. *If $\vec{p}$ is a solution of $A\vec{x} = \vec{b}$, and $\vec{v_h}$ is a solution of $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{v_h}$ is a solution of $A\vec{x} = \vec{b}$*

Example 51. *Let* $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 8 & 6 \\ 5 & -13 & -8 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. *Describe all solutions of $A\vec{x} = \vec{b}$.*

Sol:

$$[A|\vec{b}] = \begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 8 & 6 & -2 \\ 5 & -13 & -8 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 14 & 4 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 14s \\ 1 - 6s \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -14 \\ -6 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

2.3 Matrix multiplication

Let $A = [a_{ij}]_{m\times r}$ and $B = [b_{ij}]_{r\times n}$. Then

$$AB = [c_{ij}]_{m\times n},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Example 52.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 1(7)+2(9)+3(11) & 1(8)+2(10)+3(12) \\ 4(7)+5(9)+6(11) & 4(8)+5(10)+6(12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Remark. In order to have the product AB of two matrices A and B, the number of columns of A must equal the number of rows of B. So, if A is an $m \times r$ and B is an $s \times n$ matrix, in order to have the product AB, we need r = s. The resulting matrix AB will be an $m \times n$ matrix.

Properties of matrix multiplication: Let A, B, C be matrices for which sums and products are defined.

1. A(BC) = (AB)C (associativity)
2. A(B+C)= AB + AC (Left distributivity)
3. (B+C)A = BA + CA (Right distributivity)
4. r(AB) = (rA)B = A(rB)
5. Let A be $m \times n$, then $I_m A = A = AI_n$.
6. $AB \neq BA$.
7. $(AB)^T = B^T A^T$.

Powers of a matrix $A^k = AAA\cdots A$.

Example 53. *Let* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. *Compute* A^2, A^3, A^4, *then predict* A^{2012}.

Solution:

$$A^2 = AA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix},$$

$$A^3 = A^2A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix},$$

$$A^4 = A^3A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

Therefore $A^{2012} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2012 & 1 & 0 \\ 0 & -2012 & 0 & 1 \end{bmatrix}$.

Remark. A **and** B **commute if** $AB = BA$.

Example 54. *Find all 2 by 2 matrices that are commutative with* $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be commutative with B. Then $a = c + d$, $b = 0$.

Block multiplication:

It is often useful to partition a matrix into blocks: entries are themselves matrices. For example,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \Rightarrow AB = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}.$$

Example 55. *Using block multiplication to calculate* AB*, where* $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$,

$$B = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

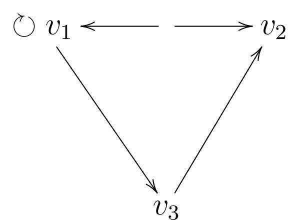
Directed Graphs: A directed graph consists of a set of points (called vertices) connected by arrows (called edges or directed arcs).

Let a graph have n vertices v_1, ..., v_n. The adjacency matrix $A = [a_{ij}]$ is the matrix whose (i, j)-entry a_{ij} is 1 if there is an edge from v_j to v_i, and zero otherwise.

A path of length r (or an r-path) from v_j to v_i is a sequence of r edges from v_j to v_i.

Theorem 8. *Let A be the adjacency matrix. Then the (i, j)-entry of A^r is the number of r-paths from v_j to v_i.*

Example 56. *Consider the graph*



(a) The adjacency matrix is: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

(b) Find a 5-path from v_1 to v_2 is: $v_1 \to v_3 \to v_2 \to v_1 \to v_1 \to v_2$.

(c) Find the number of 3-paths from v_1 to v_3.

Solution: Since $A^3 = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. The $(3, 1)$-entry is 2, which is the number of 3-paths from v_1 to v_3.

2.4 Matrix Inverses

Definition 10. *Given an $n \times n$ matrix A, the inverse of A is an $n \times n$ matrix B such that*

$$BA = AB = I,$$

where I is the $n \times n$ identity. The inverse of A is denoted by A^{-1}.

Example 57. *The inverse of 2×2 matrix: If $ad - bc \neq 0$, then*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem 9. *If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \Re^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

Properties of inverses:

1. If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
4. If A is invertible then A^k is invertible and $(A^k)^{-1} = (A^{-1})^k$.
5. If A is invertible then cA is invertible and $(cA)^{-1} = (1/c)A^{-1}$.
6. $I^{-1} = I$.
7. A triangular matrix is invertible if and only if no entry on the main diagonal is zero.
8. The inverse of an upper (lower) triangular matrix is also an upper (lower) triangular matrix.

True/False Let A, B and C denote 2×2 matrices, where A is invertible. If $AB = AC$, then $B = C$.

Solution: T

Matrix Inversion Algorithm:

$$[A|I] \xrightarrow{\textit{elementary row operations}} [I|A^{-1}].$$

Example 58. *Find A^{-1}, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}.$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Example 59. *Given the matrix equation*

$$A(B^{-1} + DX)C^T = I,$$

where A, B, C, D and X are $n \times n$ invertible matrices. Solve for X in terms of A, B, C, D.

Solution:

$$X = D^{-1}[A^{-1}(C^T)^{-1} - B^{-1}].$$

Conditions for invertibility

Theorem 10. *(The Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is an invertible matrix.*
2. *A is row equivalent to the identity matrix.*
3. *The rank of A is n.*
4. *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^n$.*
6. *There is an $n \times n$ matrix D such that $AD = I$.*
7. *A^T is invertible.*

5.4 Rank of a Matrix

Rank of a matrix is the number of leading ones in its echelon form.

Definition 11. *Let* $A = [\vec{a}_1 \ldots \vec{a_n}]_{m\times n}$.

- *The columns of* A *span (or generate) a subspace of* $\mathbb{R}^m$ *called the column space of* A*, and is denoted by* $colA$*:*
$$colA = Span\{\vec{a}_1, \ldots, \vec{a_n}\}.$$
- *The rows of* A *span (or generate) a subspace of* $\mathbb{R}^n$ *called the row space of* A*, and is denoted by* $rowA$*.*
- *The null space and image space of* A*:*
$$null\ A = \{X \in \mathbb{R}^n | AX = 0\}, \quad im\ A = \{Y \in \mathbb{R}^m | Y = AX, X \in \mathbb{R}^n\}.$$
They are subspaces of $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively.*

Example 60. *Let* A *be* $n \times n$*. Let* λ *be a number. Define*
$$E_\lambda(A) = \{X \in \mathbb{R}^n | AX = \lambda X\},$$
which is a subspace of $\mathbb{R}^n$*.*

Proof. $E_\lambda(A) = null(\lambda I - A)$.

The Rank Theorem: $dim(col\ A) = dim(row\ A) = rank\ A$.

Theorem 11. *Let* A *be* $m \times n$*. If* A *is row reduced to a row-echelon matrix* B*, then the nonzero rows of* B *form a basis of* $rowA$*. Moreover, if the columns* $j_1, \ldots, j_k$ *in* B *contain leading 1's, then the corresponding columns in* A *form a basis of* $colA$*.*

Example 61. *Let* $A = \begin{bmatrix} 1 & -3 & 1 & 12 & 7 \\ 0 & 0 & 1 & 7 & 4 \\ -2 & 6 & 0 & -10 & -6 \end{bmatrix}$.

Find a basis for $rowA$*,* $colA$ *and* $nullA$*.*

Solution: $A \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $\{[1\ -3\ 0\ 5\ 3], [0\ 0\ 1\ 7\ 4]\}$ is a basis of $rowA$;

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis of } colA.$$

To find a basis for $nullA$, consider $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0.$

$$Augmented\ matrix \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 - 3x_5 \\ x_2 \\ -7x_4 - 4x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, a basis of $nullA$ is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Theorem 12. *Let A be a square $m \times n$ matrix. Then the following statements are equivalent.*

1. *$rankA = n$*
2. *The rows of A spans $\mathbb{R}^n$.*
3. *The equation $AX = 0$ has only the trivial solution.*
4. *The columns of A are linearly independent.*
5. *$A^T A$ is invertible.*

Theorem 13. *Let A be a square $m \times n$ matrix. Then the following statements are equivalent.*

1. *$rank A = m$*
2. *The equation $AX = B$ has a solutionis consistent for every $B \in \mathbb{R}^m$.*
3. *The columns of A spans $\mathbb{R}^m$.*
4. *The rows of A are linearly independent.*
5. *AA^T is invertible.*

Theorem 14. *Let A be $m \times n$. Then $col\ A = im\ A$. $\dim(imA) + \dim(nullA) = n$.*

Proof. (1) "$col\ A \subset im\ A$": Let $A = [\vec{a_1} \ \cdots \ \vec{a_n}]$. Then $\vec{a_i} = A\vec{e_i} \in im\ A$.
(2) "$im\ A \subset col\ A$": $A\vec{x} = x_1\vec{a_1} + \cdots + x_n\vec{a_n} \in col\ A$.

5.3 Orthogonality

Dot Product, Length, and Distance

Definition 12. *Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$. i.e.*

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The number $\vec{u}^T\vec{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$ is called the inner product or dot product of $\vec{u}$ and $\vec{v}$ and is denoted as $\vec{u} \cdot \vec{v}$.

Definition 13. *The length or norm of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is a defined by,*

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + ... + v_n^2}.$$

If c is a scalar, then $||c\vec{v}|| = |c|||\vec{v}||$.

Normalization: When the length of a vector is 1 that vector is called a unit vector. Given any vector $\vec{v}$ we can change it into a unit vector:

$$\frac{\vec{v}}{||\vec{v}||}$$

is a unit vector in the direction of $\vec{v}$;

$$-\frac{\vec{v}}{||\vec{v}||}$$

is a unit vector in the opposite direction of $\vec{v}$.

This process of constructing a unit vector in the direction of a given vector $\vec{v}$ is called normalizing $\vec{v}$.

Definition 14. *Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$. The distance between $\vec{u}, \vec{v}$ is the length of the vector $\vec{u} - \vec{v}$:*

$$d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||.$$

Example 62. *Given $||X|| = 3$, $||Y|| = 2$, $X \cdot Y = 2$. Calculate the distance $d(X + Y, 3Y)$.*

Solution:

$$d(X+Y,3Y) = ||X+Y-3Y|| = \sqrt{||X-2Y||^2} = \sqrt{(X-2Y)\cdot(X-2Y)}$$
$$= \sqrt{X\cdot X - 4X\cdot Y + 4Y\cdot Y} = \sqrt{(3)^2 - 4(2) + 4(2)^2} = \sqrt{17}.$$

Angle between two vectors: Let $\vec{u}, \vec{v}$ be two vectors, and let $\theta \le \pi$ be the angle between them. Then
$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}.$$

Orthogonal Sets and Expansion Theorem

Definition 15. *(Orthogonality) Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$. They are said to be orthogonal if $\vec{u}\cdot\vec{v} = 0$.*

Theorem 15. *(The Pythagorean Theorem) Two vectors $\vec{u}, \vec{v}$ are orthogonal if and only if*
$$||\vec{u}+\vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2.$$

Definition 16. *If each pair of distinct vectors in a set is orthogonal then the set is called an orthogonal set.*

Example 63. *Let* $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$

(a) Show that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3,\}$ is an orthogonal set.

(b) Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. *Find a, b, c, d such that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{x}\}$ is orthogonal.*

Sol: (a) We need to check

$$\begin{aligned} \vec{v_1}\cdot\vec{v_2} &= 0+0+(-2)+2 = 0 \\ \vec{v_1}\cdot\vec{v_3} &= 0+(-5)+4+1 = 0 \\ \vec{v_2}\cdot\vec{v_3} &= 0+0+(-2)+2 = 0. \end{aligned}$$

Thus $\vec{v_1}\perp\vec{v}_2$, $\vec{v_1}\perp\vec{v}_3$, $\vec{v_2}\perp\vec{v}_3$, the set is orthogonal.

Theorem 16. *If a set of non-zero vectors is orthogonal, then the set is linearly independent.*

Definition 17. *(Orthogonal Basis) If* $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ *is an orthogonal set of non-zero vectors, then it is called an orthogonal basis for the subspace* $W = Span\{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$.

Example 64. *Let* $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$.

Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ *is an orthogonal basis for* $\mathbb{R}^3$.

Proof.

$$\begin{aligned} \vec{v_1} \cdot \vec{v}_2 &= 0 + (-2) + 2 = 0 \\ \vec{v_1} \cdot \vec{v}_3 &= (-5) + 4 + 1 = 0 \\ \vec{v_2} \cdot \vec{v}_3 &= 0 + (-2) + 2 = 0. \end{aligned}$$

Thus $\vec{v_1} \perp \vec{v}_2$, $\vec{v_1} \perp \vec{v}_3$, $\vec{v_2} \perp \vec{v}_3$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal of non-zero vectors, so they are linearly independent. Three such vectors automatically form a basis for $\mathbb{R}^3$.

Definition 18. *If* $S = \{\vec{u_1}, \vec{u_2}, ..., \vec{u_m}\}$ *is an orthogonal set of unit vectors, then it is called an orthonormal set. If an orthonormal set S spans some subspace W, then S is called an orthonormal basis for W. (S is an orthogonal set that spans W, so is linearly independent and thus a basis for W.)*

Example 65. *The set* $\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ *is an orthonormal set that spans* $\mathbb{R}^n$, *thus an orthonormal basis for* $\mathbb{R}^n$, *which is called the standard basis for* $\mathbb{R}^n$.

Theorem 17. *(Expansion Theorem) If* $\vec{y}$ *is a vector in a subspace* W *which has orthogonal basis* $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$, *then it may be written uniquely as a linear combination of the vectors in S,*

$$\vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + ... + c_m\vec{v}_m, \quad c_k = \frac{\vec{y} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}, k = 1, ..., m.$$

Proof.

$$\begin{aligned} \vec{y} \cdot \vec{v_1} &= (c_1\vec{v_1} + c_2\vec{v_2} + ... + c_m\vec{v}_m) \cdot \vec{v_1} \\ &= c_1\vec{v_1} \cdot \vec{v_1} + c_2\vec{v_2} \cdot \vec{v_1} + ... + c_m\vec{v_m} \cdot \vec{v_1} \\ &= c_1\vec{v_1} \cdot \vec{v_1} + 0. \end{aligned}$$

Example 66. *Let* $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}. \vec{x} = \begin{bmatrix} -3 \\ 4 \\ -1 \\ -4 \end{bmatrix}$. *Represent* $\vec{x}$ *as a linear combination of* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

8.1 Orthogonal Complements and Projections

1 Orthogonal Complements

Definition 19. *(Orthogonal Complement) Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W is defined as*

$$W^{\perp} = \{v \in \mathbb{R}^n | v \cdot w = 0, \forall w \in W\}.$$

Example 67. *Let W be a plane and L a line intersecting W. At the point of intersection of the line L to the plane W, L is orthogonal to W. We call L the orthogonal complement of W and denote it by $L = W^{\perp}$. Similarly, we may think of W as being perpendicular to L and so may be called the orthogonal complement of L and is denoted by, $W = L^{\perp}$.*

Properties of Orthogonal Complement:

- A vector $\vec{x}$ is in $W^{\perp}$ if and only if $\vec{x}$ is orthogonal to every vector in a set that spans W.
- $W^{\perp}$ is a subspace.
- $W \bigcap W^{\perp} = \{0\}$.
- If $W = span\{w_1, ..., w_k\}$, then $v \in W^{\perp}$ if and only if $v \cdot w_i = 0$ for all i.
- $(rowA)^{\perp} = nullA$, $(colA)^{\perp} = nullA^T$.

Example 68. *Let*

$$W = span\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}, \quad Y = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}.$$

1) Show that $Y \in W^{\perp}$.
2) Find other vectors in $W^{\perp}$.

Solution: 1)

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot Y = 0, \quad \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot Y = 0.$$

2) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W^{\perp}$. Then

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \vec{x} = 0, \qquad \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{x} = 0 \Rightarrow$$

$$A\vec{x} = 0, \quad A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/7 \\ 0 & 1 & 4/7 \end{bmatrix}.$$

The solution of this is: $x_1 = (-1/7)x_3, x_3 = (-4/7)x_3$, i.e.,

$$\vec{x} = s \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}.$$

2 Projections

Definition 20. *(Orthogonal projection) Let W be a subspace of $\mathbb{R}^n$. Let $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ be an orthogonal basis of W. Then the orthogonal projection of $\vec{y}$ onto W is defined as*

$$\operatorname{proj}_W \vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + ... + c_m\vec{v}_m, \quad c_k = \frac{\vec{y} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}, k = 1, ..., m.$$

The complement of $\vec{y}$ orthogonal to W is

$$perp_W\vec{y} = \vec{y} - \operatorname{proj}_W \vec{y}.$$

Theorem 18. *(The orthogonal Decomposition Theorem) Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely in the form*

$$\vec{y} = Y_1 + Y_2, \quad Y_1 \in W, Y_2 \in W^{\perp},$$

where

$$Y_1 = \operatorname{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v_1} + \cdots + \frac{\vec{y} \cdot \vec{v}_m}{\vec{v}_m \cdot \vec{v}_m}\vec{v_m}, \quad Y_2 = \vec{y} - Y_1.$$

Example 69. *Let* $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ -5 \\ -2 \\ 1 \end{bmatrix}$. *Let* $W = Span\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. *Find $Y_1 \in W, Y_2 \in W^{\perp}$ such that $\vec{y} = Y_1 + Y_2$.*

Solution: Since

$$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= 0+0+(-2)+2=0 \\ \vec{v_1} \cdot \vec{v_3} &= 0+(-5)+4+1=0 \\ \vec{v_2} \cdot \vec{v_3} &= 0+0+(-2)+2=0. \end{aligned}$$

Thus $\vec{v_1} \perp \vec{v_2}$, $\vec{v_1} \perp \vec{v_3}$, $\vec{v_2} \perp \vec{v_3}$, the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is orthogonal basis of W.

$$Y_1 = \operatorname{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v_1} + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v_2} + \frac{\vec{y} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v_3} = 0 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \frac{-6}{5} \begin{bmatrix} 0 \\ -5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \\ 0 \end{bmatrix}.$$

$$Y_2 = \vec{y} - Y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -2 \\ 1 \end{bmatrix}.$$

Property: If $\vec{y} \in W$, then $\operatorname{proj}_W \vec{y} = \vec{y}$.

Proof. Let $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ be an orthogonal basis for W. Since $\vec{y} \in W$,

$$\vec{y} = d_1\vec{v_1} + d_2\vec{v_2} + ... + d_m\vec{v_m}. \Rightarrow$$

$$\vec{y} \cdot \vec{v_1} = d_1\vec{v_1} \cdot \vec{v_1}, ..., \vec{y} \cdot \vec{v_m} = d_m\vec{v_m} \cdot \vec{v_m}.$$

Thus

$$\begin{aligned} \operatorname{proj}_W \vec{y} &= \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v_1} + \cdots + \frac{\vec{y} \cdot \vec{v}_m}{\vec{v}_m \cdot \vec{v}_m}\vec{v_m} \\ &= \frac{d_1\vec{v_1} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\vec{v_1} + \cdots + \frac{d_m\vec{v_m} \cdot \vec{v_m}}{\vec{v_m} \cdot \vec{v_m}}\vec{v_m} \\ &= d_1\vec{v_1} + d_2\vec{v_2} + ... + d_m\vec{v_m} \\ &= \vec{y}. \end{aligned}$$

Theorem 19. *Let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_m}\}$. Then for each $\vec{y}$ in $\mathbb{R}^n$,*

$$\operatorname{proj}_W \vec{y} = (\vec{y} \cdot \vec{u}_1)\vec{u_1} + \cdots + (\vec{y} \cdot \vec{u}_m)\vec{u_m}.$$

Theorem 20. *If W is a subspace of $\mathbb{R}^n$, then*

$$\dim W + \dim W^{\perp} = n.$$

Proof. (1) A basis of W and a basis of $W^{\perp}$ form a linearly independent set; (2) By decomposition theorem, any vector in $\mathbb{R}^n$ can be written as a linear combination of the set.

3 Approximation

Theorem 21. *(The Best Approximation Theorem) Let W be a subspace of $\mathbb{R}^n$, $\vec{y}$ in $\mathbb{R}^n$ and $\hat{\vec{y}}$ be the orthogonal projection of $\vec{y}$ onto W. Then $\hat{\vec{y}}$ is the closest point in W to $\vec{y}$ in the sense that*

$$||\vec{y} - \hat{\vec{y}}|| < ||\vec{y} - \vec{v}||$$

for all $\vec{v} \in W$ distinct from $\hat{\vec{y}}$.

Example 70. *Let* $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$. *Let $W = Span\{\vec{v}_1, \vec{v}_2\}$. Find the distance from $\vec{y}$ to W.*

Solution: The closest point in W to $\vec{y}$ is $proj_W\vec{y}$. So the distance is $||\vec{y} - proj_W\vec{y}||$. Note that $\vec{v_1} \cdot \vec{v_2} = 0$, $\{\vec{v}_1, \vec{v}_2\}$ is orthogonal basis of W,

$$proj_W\vec{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v_1} + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v_2} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}, \Rightarrow \vec{y} - proj_W\vec{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \Rightarrow ||\vec{y} - proj_W\vec{y}|| = 8.$$

4 The Gram-Schmidt Algorithm

Let $S = \{X_1,\ X_2, \cdots,\ X_k\}$ be a set of vectors which are linearly independent, and let

$$\begin{aligned} F_1 &= X_1 \\ F_2 &= X_2 - \frac{X_2 \cdot F_1}{||F_1||^2}F_1 \\ &\cdots \\ F_k &= X_k - \frac{X_k \cdot F_1}{||F_1||^2}F_1 - \frac{X_k \cdot F_2}{||F_2||^2}F_2 - \cdots - -\frac{X_k \cdot F_{k-1}}{||F_{k-1}||^2}F_{k-1}. \end{aligned}$$

Then $\{F_1,\ F_2, \cdots,\ F_k\}$ is an orthogonal set.

Example 71. *Consider the following independent set* $S = \{X_1,\ X_2,\ X_3\}$ *of vectors from* $\mathbb{R}^4$*:*

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 4 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to convert the set $S = \{X_1,\ X_2,\ X_3\}$ *into an orthogonal set* $B = \{F_1,\ F_2,\ F_3\}$*.*

Solution: Let $F_1 = X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{||F_1||^2} F_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}.$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{||F_1||^2} F_1 - \frac{X_3 \cdot F_2}{||F_2||^2} F_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 4 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-6}{24} \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \\ 0.5 \\ 3 \end{bmatrix}.$$

3.1 Cofactor Expansion

Definition 21. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The determinant of* A *is defined as*

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a $n \times n$ *matrix* A, *let* A_{ij} *be the matrix obtained from* A *by deleting the i-th row and j-th column. The* $(i,j)^{th}$ *cofactor of* A *is the number,*

$$c_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + ... + a_{in}c_{in},$$

which is called a cofactor expansion across the i-th row. Similarly,

$$\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + ... + a_{nj}c_{nj},$$

which is called a cofactor expansion across the j-th column.

Example 72. *Calculate* $\det A$, *where*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

Solution: We do cofactor expansion across the 2nd row.

$$\begin{aligned} \det A &= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= 2(-1)^{2+1} \det \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} + (-1)^{2+2} \det \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} + (-1)^{2+3} \det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \\ &= 2(14) + (-13) + 5 = 20. \end{aligned}$$

Geometric Interpretation:

If A is 2×2, then $|A|$ represents the area of the parallelogram formed by the two row vectors. If A is 3×3, then $|A|$ represents the volume of the parallelepiped formed by the three row vectors.

Definition 22. *A triangular matrix is a matrix that is all zeros either above or below the diagonal. An upper triangular matrix means all entries below the main diagonal are zero; an lower triangular matrix means all entries above the main diagonal are zero.*

Theorem 22. *If A is a triangular matrix then $\det A$ is the product of the entries on the main diagonal of A.*

Example 73. *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 5 & 3 & 5 & 7 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

Solution: A is an upper triangular matrix. $\det A = 5(1)(2)(12) = 120$.

Theorem 23. *If A is a 3×3 matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then*

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Remark. This comes from the three main diagonals and three other diagonals by repeating the first two columns.

Elementary Operations and Determinants

1. If $A \xrightarrow{R_i \to R_i + kR_j} B$, then $\det B = \det A$.

2. If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det B = -\det A$.

3. If $A \xrightarrow{R_i \to kR_i} B$, then $\det B = k \det A$.

Example 74. *Let*

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -4 & 12 & -4 & 5 \\ 2 & -5 & 4 & -3 \\ -3 & 10 & -1 & 7 \end{bmatrix}.$$

(a) Calculate $\det A$ *by using row reduction. (b) Find* C_{23}*, the (2,3)-cofactor of* A*.*

Solution.

$$\det A = \begin{vmatrix} 1 & -3 & 2 & -4 \\ -4 & 12 & -4 & 5 \\ 2 & -5 & 4 & -3 \\ -3 & 10 & -1 & 7 \end{vmatrix} \xlongequal[R_4 \to R_4 + 3R_1]{\substack{R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - 2R_1}} \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 4 & -11 \\ 0 & 1 & 0 & 5 \\ 0 & 1 & 5 & -5 \end{vmatrix}$$

$$\xlongequal{R_2 \leftrightarrow R_4} - \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 4 & -11 \end{vmatrix} \xlongequal{R_3 \to R_3 - R_2} - \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & -5 & 10 \\ 0 & 0 & 4 & -11 \end{vmatrix}$$

$$\xlongequal{R_3 \to -\frac{1}{5} R_3} - (-5) \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & -11 \end{vmatrix} \xlongequal{R_4 \to R_4 - 4R_3} 5 \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{vmatrix}$$

$$= 5(1)(1)(-3) = -15.$$

(b)

$$C_{23} = (-1)^{2+3} \det A_{23} = -\det \begin{bmatrix} 1 & -3 & -4 \\ 2 & -5 & -3 \\ -3 & 10 & 7 \end{bmatrix}$$

$$\xlongequal{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1}} - \begin{vmatrix} 1 & -3 & -4 \\ 0 & 1 & 5 \\ 0 & 1 & -5 \end{vmatrix} \xlongequal{R_3 \to R_3 - R_2} - \begin{vmatrix} 1 & -3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & -10 \end{vmatrix} = -1(1)(-10) = 10.$$

Example 75. *Given that* $\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, *what is* $\begin{vmatrix} a-2g & b-2h & c-2i \\ 3g & 3h & 3i \\ 5d & 5e & 5f \end{vmatrix}$*?*

Solution. $\begin{vmatrix} a-2g & b-2h & c-2i \\ 3g & 3h & 3i \\ 5d & 5e & 5f \end{vmatrix}$ = -15$\begin{vmatrix} a-2g & b-2h & c-2i \\ d & e & f \\ g & h & i \end{vmatrix}$

= -15$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -15(2) = -30.$

Example 76. *(Vandermonde determinant)*

$$det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{i>j;\ i,j=1}^{n} (x_i - x_j).$$

Properties of determinants:

1. If A is a square matrix then $\det A^T = \det A$.
2. If A has two identity rows (or columns), then $\det A = 0$.
3. If A is an $n \times n$ matrix and k a scalar then $\det(kA) = k^n \det A$.

3.2 Determinants and Matrix Inverses

Product Theorem:

$$\det(AB) = \det A \det B.$$

Example 77. *Let A, B and C be 3×3 invertible matrices, $det(A) = 3$, $det(B) = 5$, $det(C) = 6$. Calculate $det(A^{-1}C^2(-2B^T))$.*

Properties of determinants:

4. If A is invertible then $det(A^{-1}) = \frac{1}{\det A}$.

5. A square matrix A is invertible $\Leftrightarrow \det A \neq 0$.

6. Block upper (lower) triangular matrices:

$$\det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \det A \det B, \quad \det \begin{bmatrix} A & 0 \\ X & B \end{bmatrix} = \det A \det B.$$

7. **Inverse Formula:** If A is $n \times n$ and invertible, then

$$A^{-1} = \frac{1}{\det A} adj A, \quad adj A = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix},$$

where $adj A$ is called adjugate of A.

Proposition 5. *Let A be $n \times n$. Then* $\det(adj A) = (\det A)^{n-1}$.

A square matrix A is called **orthogonal** if $A^{-1} = A^T$.

Example 78. $A = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

3.3 Diagonalization and Eigenvalues

Eigenvalues and Eigenvectors

Definition 23. *An eigenvector of an $n \times n$ matrix A is a nonzero vector X such that $AX = \lambda X$ for some scalar λ. A scalar λ is called an eigenvalue of A, X is called the eigenvector corresponding to λ.*

To determine whether a given value λ is an eigenvalue of a matrix A we need to find a non-zero vector X such that $AX = \lambda X$. This is the same as determining whether the matrix equation

$$(\lambda I - A)X = 0$$

has a non-trivial solution.

Characteristic equation: $c_A(\lambda) = \det(\lambda I - A)$ is called the characteristic polynomial of A and

$$c_A(\lambda) = \det(\lambda I - A) = 0$$

is called the characteristic equation.

Theorem 24. *The solutions of the characteristic equation are the eigenvalues of A. The eigenvectors X corresponding to λ are the nonzero solutions of $(\lambda I - A)X = 0$.*

Basic eigenvectors: Any basic solution of $(\lambda I - A)X = 0$ is called a basic eigenvector (or basic-λ-eigenvector). Any set of non-zero multiples of a basic eigenvector of $(\lambda I - A)X = 0$ will be called a set of basic eigenvectors corresponding to λ. We treat a set of basic eigenvectors as only one basic eigenvector.

Example 79. *Let* $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. *Find all eigenvalues.*

Solution: Eigenvalues are 5, 1. Corresponding eigenvectors are $X = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $X = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Example 80. *Let* $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$. *Find the eigenvalues and eigenvectors.*

Solution:

$$\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 1 & -5 & 5-\lambda \end{vmatrix} \overset{R_3\to R_3-R_2}{=\!=\!=} \begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 0 & -3+\lambda & 3-\lambda \end{vmatrix}$$

$$=(3-\lambda)\begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 0 & -1 & 1 \end{vmatrix}=(3-\lambda)\left\{(3-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ -1 & 1 \end{vmatrix}-1\begin{vmatrix} -4 & 2 \\ -1 & 1 \end{vmatrix}\right\}$$

$$=(3-\lambda)\{(3-\lambda)(-\lambda)+2\}=(3-\lambda)(\lambda-1)(\lambda-2).$$

Thus the eigenvalues are 1, 2, 3 .

When $\lambda=3$,

$$A-3I=\begin{bmatrix} 0 & -4 & 2 \\ 1 & -5 & 2 \\ 1 & -5 & 2 \end{bmatrix} \xrightarrow{R_3\to R_3-R_2} \begin{bmatrix} 0 & -4 & 2 \\ 1 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-3I)\vec{x}=0$ has the solution

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\frac{1}{2}x_3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

We take $\vec{x}_1=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as a basic eigenvector.

When $\lambda=1$,

$$A-I=\begin{bmatrix} 2 & -4 & 2 \\ 1 & -3 & 2 \\ 1 & -5 & 4 \end{bmatrix} \xrightarrow{R_1\to \frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & 2 \\ 1 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-I)\vec{x}=0$ has the solution

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We take $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a basic eigenvector.

When $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 1 & -4 & 2 \\ 1 & -4 & 2 \\ 1 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - 2I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

We take $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ as a basic eigenvector.

Diagonalization

Multiplicity:

- The multiplicity of an eigenvalue is equal to the number of times it is a root of the characteristic equation.

Similar matrices: Two matrices A and B are similar if there is an invertible matrix P such that,

$$A = PBP^{-1}.$$

Theorem 25. *If $n \times n$ matrices A and B are similar, then they have the same characteristics polynomial and hence the same eigenvalues (with the same multiplicities).*

Proof.

$$\begin{aligned} \det(A - \lambda I) &= \det(PBP^{-1} - \lambda PP^{-1}) \\ &= \det[P(B - \lambda I)P^{-1}) \\ &= \det(P)\det(B - \lambda I)\det(P^{-1}) \\ &= \det(P)\det(B - \lambda I)\frac{1}{\det(P)} \\ &= \det(B - \lambda I). \end{aligned}$$

A diagonal matrix is a matrix with only zeros on its off diagonal entries.

Definition 24. *If an $n \times n$ matrix A is similar to a diagonal matrix D then A is said to be diagonalizable.*

Properties:

(1) Let X be an eigenvector of the matrix A corresponding to the eigenvalue a. For any positive integer n, a^n is an eigenvalue of A^n with corresponding eigenvector X.

(2) Let X be an eigenvector of the matrix A corresponding to the eigenvalue a. If A is invertible, then $\frac{1}{a}$ is an eigenvalue of A^{-1} with corresponding eigenvector X.

(3) Let X be an eigenvector of the matrix A corresponding to the eigenvalue a. If A is invertible, then for any integer n, a^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector X.

(4) Let X be an eigenvector of both the matrices A and B associated with eigenvalues a and b. X is an eigenvector of $A + B$ associated with eigenvalue $a + b$.

(5) Let X be an eigenvector of both the matrices A and B associated with eigenvalues a and b. X is an eigenvector of AB associated with eigenvalue ab.

(6) If A is invertible, then 0 can not be an eigenvalue.

Solution: Proof. (1) $A^nX = a^nX$. (2) $AX = aX \Rightarrow X = A^{-1}(aX) = aA^{-1}X$.

Theorem 26. *(Diagonalization Theorem) Let A be an $n \times n$ matrix.*

- *A is diagonalizable if and only if A has n eigenvectors $X_1, ..., X_n$ such that $P = [X_1 \ \cdots X_n]$ is invertible. When this is the case, $P^{-1}AP = diag(\lambda_1, ..., \lambda_n)$, where each λ_i is the eigenvalue corresponding to X_i.*

- *If A has n distinct eigenvalues, then A is diagonalizable.*

- *A is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the number of basic eigenvectors corresponding to λ.*

Example 81. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ *is diagonalizable: 4 distinct eigenvalues.*

$B = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ *is not diagonalizable: $\lambda = 4$, one eigenvector* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 82. *Let* $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

1) Find P and D such that $A = PDP^{-1}$.
2) Calculate A^4.

Sol: 1)

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}; \ or\ P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

2) Let $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, then $P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$.

$$A^4 = \left\{PDP^{-1}\right\}^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^4 \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 364 & 261 \\ 348 & 277 \end{bmatrix}.$$

Example 83. *Diagonalize the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Solution: Step 1: Find all eigenvalues: The characteristic polynomial is $-\lambda^3 + 6\lambda^2 - 9\lambda + 4$. So $\lambda = 1, 4$.

Step 2: Basic eigenvectors corresponding to $\lambda = 1$: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Basic eigenvectors corresponding to $\lambda = 4$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Step 3: Construct P and D:

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Applications:

- Dynamics system
- Google PageRank

A Model of Population Dynamics

Example 84. *In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 3% of the suburban population moves into the city. In 2000, there were 800,000 residents in the city and 500,000 in the suburbs.*

1) Set up a difference equation.

2) Estimate the population in the city and in the suburbs in 2002.

Solution: 1) Let $x_k = \begin{bmatrix} c_k \\ s_k \end{bmatrix}$ be the population at the time of the kth measurement ($k = 0 \leftrightarrow$ initial population in 2000), where c_k is the population in the city, s_k is the population in the suburbs, $c_0 = 800,000$, $s_0 = 500,000$. Then

$$c_1 = (1 - 7\%)c_0 + (3\%)s_0 = \begin{bmatrix} 0.93 & 0.03 \end{bmatrix} \begin{bmatrix} c_0 \\ s_0 \end{bmatrix},$$

$$s_1 = (7\%)c_0 + 1 - 3\%)s_0 = \begin{bmatrix} 0.07 & 0.97 \end{bmatrix} \begin{bmatrix} c_0 \\ s_0 \end{bmatrix},$$

which implies that

$$\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix} \begin{bmatrix} c_0 \\ s_0 \end{bmatrix}.$$

Hence

$$x_{k+1} = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix} x_k.$$

2) For 2002, $k = 2$. We have

$$x_1 = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix} x_0 = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix} \begin{bmatrix} 800000 \\ 500000 \end{bmatrix} = \begin{bmatrix} 759000 \\ 541000 \end{bmatrix},$$

$$x_2 = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix} x_1 = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix} \begin{bmatrix} 759000 \\ 541000 \end{bmatrix} = \begin{bmatrix} 722100 \\ 577900 \end{bmatrix}.$$

2.2, 2.6, 7.1, 7.2 Linear Transformation

Transformations

Definition 25. *A transformation, (function or mapping), T from vector space V to vector space W is a rule that assigns a vector $T(\vec{x}) \in W$ for each vector $\vec{x} \in V$.*

$$T : V \to W$$

V is called the domain and W the codomain of T. The vector $T(\vec{x})$ is called the image of $\vec{x}$. The set of all images of T is called the range of T.

Example 85. *$T(x, y) = (x + y, x - y, 2x + 3y)$ is a transformation from $\mathbb{R}^2 \to \mathbb{R}^3$.*

Matrix transformation induced by A: Let A be $m \times n$. Then the matrix A is a transformation from $\mathbb{R}^n \to \mathbb{R}^m$. We denote this by T_A.

Example 86. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix}$.

(i) Then the matrix $T_A = A$ is a transformation from $\mathbb{R}^3 \to \mathbb{R}^2$. Find $T_A(\vec{b})$, where $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(ii) Find $\vec{x}$ such that $T_A(\vec{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(iii) Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the range of $T_A(\vec{x})$?

Solution: (i) $T_A(\vec{b}) = A\vec{b} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

Linear Transformations:

Definition 26. *A transformation T is **linear** if*

1. *$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T.*
2. *$T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ and scalars c.*

Properties: If T is linear, then

- $T(\vec{0}) = \vec{0}$.

- $T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T and all scalars c, d.

Example 87. *T_A is linear for any matrix A.*

Example 88. *$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = \vec{x}$. Then T is linear and is called **identity transformation** I.*

Example 89. *$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = r\vec{x}$, r a scalar. Then T is linear.*

Example 90. *$T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x} + \vec{b}$, $\vec{b} \neq \vec{0}$. Then T is non-linear, since $T(\vec{0}) \neq \vec{0}$.*

Definition 27. *Two linear transformations S and T from $\mathbb{R}^n$ to $\mathbb{R}^m$ are equal if $T(\vec{x}) = S(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$.*

Theorem 27. *Let S and T be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. If for a basis $B = \{\vec{b_1}, ..., \vec{b_n}\}$, $S(\vec{b_i}) = T(\vec{b_i})$, $i = 1, ..., n$, then $S = T$.*

Matrix of a linear transformation:

Theorem 28. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^n$. A is an $m \times n$ matrix:*

$$A = [T(\vec{e}_1)...T(\vec{e}_n)].$$

*A is called the **matrix associated with** T (or the standard matrix for the linear transformation T).*

Example 91. *Let $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$ or $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$.*

1) Find the matrix associated with T.

2) Solve $T(\vec{x}) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, or find $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $T(x_1, x_2) = (0, 1, 4)$.

Solution: 1) $T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$.

2)

$$\text{augmented matrix} = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 1 \\ 3 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Example 92. *Rotations, Reflections, Projections:*

Let R_θ denote counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ. Then $R_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

Let Q_m denote the reflection in the line $y = mx$. Then $Q_m(\vec{x}) = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$.

Let P_m denote the projection on the line $y = mx$. Then $P_m(\vec{x}) = \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$.

Combination and Composition of Linear Transformations

Combinations:

- Let S and T be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, c be a scalar. Then $S \pm T$, $-S$, cS are linear, where

$$(S \pm T)(\vec{x}) = S(\vec{x}) \pm T(\vec{x}); (-S)(\vec{x}) = -S(\vec{x}); (cS)(\vec{x}) = cS(\vec{x}).$$

- Let L be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then the powers $L^0 = I$, $L^{k+1} = L(L^k)$ are linear, $k = 0, 1, 2, ...$

Combinations:

Definition 28. *Let U, V and W be vector spaces. Let $T : U \to V$ and $S : V \to W$ be two transformations. Then the composition of S with T is $S \circ T$, defined by:*

$$(S \circ T)(\vec{u}) = S(T(\vec{u})), \quad \vec{u} \in U.$$

Theorem 29. *Let S and T be linear transformations with associated matrices A and B respectively. Then*

- $(S \pm T)(\vec{x}) = (A \pm B)(\vec{x})$;
- $(cS)(\vec{x}) = cA(\vec{x})$;
- $(-S)(\vec{x}) = -A(\vec{x})$;
- $(T \circ S)(\vec{x}) = BA(\vec{x})$.

Example 93. *Let $T(x,y,z) = (x+2y-z, 3x-z, y+z)$. Find $2T^2 - T + 3I$.*

Solution: $(2T^2 - T + 3I)(\vec{x}) = (2A^2 - A + 3I)(\vec{x}) = (16x - 7z, 3x + 13y - 7z, 6x + y + 2z)$.

Example 94. *$T: M_{nn} \to \mathbb{R}$ by $T(A) = |A| = \det(A)$. Then T is non-linear.*

Example 95. *Let $T: M_{22} \to M_{22}$ be given by $T(A) = A^T - A$. Then T is linear.*

Example 96. *Let $T: \mathbb{R}^2 \to P_3$ by $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = 1 - x - x^2$, and $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = 2x + x^3$. Find $T(\begin{bmatrix} 5 \\ 4 \end{bmatrix})$ and $T(\begin{bmatrix} a \\ b \end{bmatrix})$.*

Solution: Note that $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus

$$T(\begin{bmatrix} 5 \\ 4 \end{bmatrix}) = 2(1 - x - x^2) + 1(2x + x^3) = 1 - 2x^2 + x^3.$$

Theorem 30. *If T and S are linear, then $S \circ T$ is linear.*

Example 97. *Let $T: \mathbb{R}^2 \to M_{22}$ and $S: M_{22} \to P_3$ be two linear transformations defined by:*

$$T(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix}, \quad S(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = a + (b+c)x - dx^2 + (a+c)x^3.$$

Find $S \circ T$.

Kernel and Image

Definition 29. *Let $T: V \to W$ be a linear transformation. The kernel (or nullspace) of T:*

$$ker(T) = \{\vec{v} \in V | T(\vec{v}) = \vec{0}\};$$

The image (or range) of T:

$$im(T) = \{T(\vec{v}) | \vec{v} \in V\}.$$

Theorem 31. *If $TX = AX$, then $ker(T) = null(A)$, $im(T) = im(A) = col(A)$.*

Example 98. *(i) If $D : P_3 \to P_2$, then $ker(D) = \mathbb{R}$, $im(D) = P_2$.*

(ii) If $S : P_1 \to \mathbb{R}$ is given by $S(p(x)) = \int_0^1 p(x)dx$, then

$$ker(S) = \{-\frac{b}{2} + bx | b \in \mathbb{R}\}, \quad im(S) = \mathbb{R}.$$

(iiii) If $T : M_{22} \to M_{22}$ is given by $T(A) = A^T$, then $ker(T) = \{0\}$, $im(T) = M_{22}$.

Example 99. *Let* $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+2z \\ y+3z \\ -2x-4z \\ -y-3z \end{bmatrix}$ *or*

$T(x, y, z) = (x + 2z, y + 3z, -2x - 4z, -y - 3z)$.

1) Find a basis, dimension, geometric meaning for ker(T).

2) Find a basis, dimension, geometric meaning for im(T).

Solution: $\ker T = \{s \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} | s \in \mathbb{R}\}$. $\operatorname{im} T = \{s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} | s, t \in \mathbb{R}\}$.

One-to-One, Inverse and Onto

Definition 30. *Let $T : V \to W$ be a linear transformation. If T maps distinct vectors in V to distinct vectors in W, then T is called one-to-one. If $im(T) = W$, then T is called onto.*

Theorem 32. *Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then,*

1. T_A is onto if and only if the columns of A span $\mathbb{R}^m$.

2. T_A is 1:1 if and only if the columns of A are linearly independent.

3. T_A is 1:1 if and only if $A\vec{x} = T_A(\vec{x}) = 0$ has only the trivial solution.

Example 100. *Let* $A = \begin{bmatrix} 1 & 0 & 9 \\ 0 & 3 & 7 \end{bmatrix}$. *Is $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ ONTO, 1:1?*

Solution: T_A is ONTO, since the columns of A span $\mathbb{R}^2$. T_A is not 1:1, since the columns of A are linearly dependent.

Example 101. *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by*

$$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Is T ONTO, 1:1?

Solution: T is not ONTO, since A has at most two pivots, the columns of A can not span $\mathbb{R}^3$. T_A is 1:1, since the columns of A are linearly independent.

Theorem 33. *A linear transformation $T : V \to W$ is one-to-one iff $ker(T) = \{0\}$.*

Proof. If $T(u) = T(v)$, then $T(u - v) = 0$.

Theorem 34. *Let $\dim V = \dim W = n$. A linear transformation $T : V \to W$ is one-to-one if and only if it is onto.*

Proof. " $\Rightarrow$ ": Let T be 1:1, then $ker(T) = \{0\}$. Let $B_V = \{v_1, ..., v_n\}$. Then $\{T(v_1), ..., T(v_n)\}$ is a basis of range(T). By Rank Theorem, rank(T)=dim(V)=dim(W). Thus $\{T(v_1), ..., T(v_n)\}$ is a basis of W.

" $\Leftarrow$ " $rank(T) = n$, thus $nullity(T) = 0$. Thus 1:1.

Example 102. *Let $T : \mathbb{R}^2 \to P_1$ be defined by:*

$$T(\begin{bmatrix} a \\ b \end{bmatrix}) = a - b + (b + a)x.$$

Show that T is onto and one-to-one.

Inverse:

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix A. If A is invertible, then the inverse of T is given by: $T^{-1}(\vec{x}) = A^{-1}(\vec{x})$.

Theorem 35. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix A. Then the following are equivalent:*

1. *T is one-to-one,*
2. *T is onto,*
3. *T is invertible,*
4. $\det(A) \neq 0$.

Proof. " $\Rightarrow$ ": Let T be 1:1, then $ker(T) = \{0\}$. Let $B = \{v_1, ..., v_n\}$. Then $\{T(v_1), ..., T(v_n)\}$ is a basis of range(T). By Rank Theorem, $rank(T) = dim(\mathbb{R}^n)$. Thus $\{T(v_1), ..., T(v_n)\}$ is a basis of $\mathbb{R}^n$.

" $\Leftarrow$ " $rank(T) = n$, thus $nullity(T) = 0$. Thus 1:1.

Dimension

Dimension Theorem (Rank Theorem): Let $T : V \to W$ be a linear transformation. Then

$$\dim(kerT) + \dim(imT) = \dim V.$$

Proof. Let $\dim V = n$, and let $\{v_1, \cdots, v_k\}$ be a basis for $ker(T)$. Then we can extend it to a basis of V: $\{v_1, \cdots, v_k, v_{k+1}, \cdots v_n\}$. We only need to prove that $\{T(v_{k+1}), \cdots T(v_n)\}$ is a basis for $range(T)$.

Example 103. *Let $T : \mathbb{R}^2 \to M_{22}$ and $S : M_{22} \to P_3$ be two linear transformations defined by:*

$$T(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix}, \quad S(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = a - b + (b+c)x + (a-d)x^2 + (a+c)x^3.$$

Find $\dim(imT), \dim(imS), \dim(kerT), \dim(kerS)$.