SOL: -A-

1. [3 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{\sin^2 x \arctan x}{(x+2)^4}$.

NOTE:

$$\ln(f(x)) = \ln\left(\frac{\sin^2 x \cdot \arctan x}{(x+2)^4}\right) =$$

$$= 2\ln(\sin x) + \ln(\arctan x) - 4\ln(x+2).$$

SO:

$$\frac{d}{dx}\left(\ln(f(x))\right) = 2 \cdot \frac{\cos x}{\sin x} + \frac{\frac{1}{1+x^2}}{\arctan x} - 4 \cdot \frac{1}{x+2}$$

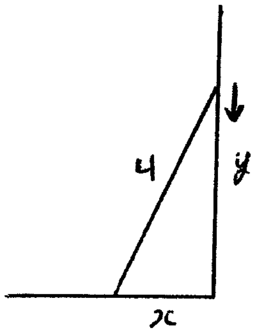
BUT

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}.$$

$\underline{\text{SO}}$: $$f'(x) = \frac{\sin^2 x \cdot \arctan x}{(x+2)^4}\left[\frac{2\cos x}{\sin x} + \frac{\frac{1}{1+x^2}}{\arctan x} - \frac{4}{x+2}\right]$$

A

2. [3 points] A ladder 4 m long is leaning against a wall. The top of the ladder is sliding down the wall at 0.25 m/s. How fast is the bottom of the ladder sliding along the floor when the ladder is 2 m from the wall?



let y be the height of the ladder up the wall and x the distance between the ladder and the wall

then we're told that $\frac{dy}{dt} = -0.25$ m/s

and we want $\frac{dx}{dt}$ when $x = 2$ m

by Pythagoras, $x^2 + y^2 = 4^2 = 16$

so $\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(16)$

or $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ or $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

when $x = 2$ m, $y = \sqrt{16-(2)^2} = \sqrt{12}$ m $= 2\sqrt{3}$ m

so $\frac{dx}{dt} = -\frac{2\sqrt{3}}{2}(-0.25 \text{ m/s}) = \boxed{\frac{\sqrt{3}}{4} \text{ m/s}} \approx \boxed{0.4330 \text{ m/s}}$

3. [2 points] Find the absolute maximum and minimum of $f(x) = x - 2\ln x$ on $[1, 6]$.

$$f'(x) = 1 - \frac{2}{x} = \frac{x-2}{x}$$

$$f'(x) = 0 \iff x = 2 \in [1, 6].$$

$$f(1) = 1 - 2\ln(1) = 1 - 0 = 1$$

$$f(2) = 2 - 2\ln(2) \cong 0.6137$$

$$f(6) = 6 - 2\ln(6) \cong 2.416$$

SO: MAX is $6 - 2\ln 6$ (at $x=6$)

MIN is $2 - 2\ln 2$ (AT $x=2$)

4. [4 points] Find the following limits:

(a) $\lim_{x\to 0^+} \dfrac{e^x - \cos x - x}{2010x^2}$ (b) $\lim_{x\to\infty} \left(1 - \dfrac{4}{x}\right)^x$

a) $\lim_{x\to 0^+} \dfrac{e^x - \cos x - x}{2010x^2}$ it is $\frac{0}{0}$ (L'H);

$= \lim_{x\to 0^+} \dfrac{e^x + \sin x - 1}{4020x}$ it is $\frac{0}{0}$ (L'H);

$= \lim_{x\to 0^+} \dfrac{e^x + \cos x}{4020} = \dfrac{2}{4020} = \boxed{\dfrac{1}{2010}}$

b) $\lim_{x\to\infty} \left(1 - \dfrac{4}{x}\right)^x$ it is 1^∞;

Set $y = \left(1 - \dfrac{4}{x}\right)^x$; THEN $\ln y = x \ln\left(1 - \frac{4}{x}\right)$.

Note: $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 - \frac{4}{x}\right) = \lim_{x\to\infty} \dfrac{\ln\left(1 - \frac{4}{x}\right)}{x^{-1}}$ $\left[\frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{x\to\infty} \dfrac{\frac{1}{1-\frac{4}{x}} \cdot \left(\frac{4}{x^2}\right)}{-x^{-2}} = \lim_{x\to\infty} \dfrac{-4}{1 - \frac{4}{x}} = -4.$

Hence: $\ln\left(\lim_{x\to\infty} y\right) = -4$; THUS

$\lim_{x\to\infty} y = e^{-4}$ OR: $\lim_{x\to\infty} \left(1 - \frac{4}{x}\right)^x = e^{-4}$

(Recall how we defined y!!!)

5. [2 points] If $f''(x) = 4e^{2x} + 6x^2 - 3\sin x + 1$, find the most general f.

$\Downarrow$

$$f'(x) = \frac{4}{2}e^{2x} + \frac{6}{3}x^3 + 3\cos x + x + C \quad ; \quad C\#$$

$$= 2e^{2x} + 2x^3 + 3\cos x + x + C ;$$

$\Downarrow$

$$f(x) = \frac{2}{2}e^{2x} + \frac{2}{4}x^4 + 3\sin(x) + \frac{x^2}{2} + Cx + D ;$$

$D\#$

i.e.

$$f(x) = e^{2x} + \frac{1}{2}x^4 + 3\sin(x) + \frac{x^2}{2} + Cx + D$$

C, D #s

A

6. [6 points] Consider the function $y = f(x) = \frac{x}{x+2}$.

(i) Find any vertical or horizontal asymptotes.
(ii) Find the intervals of increase and decrease and any local extrema.
(iii) Find the intervals of concavity and any inflection points.
(iv) Use all of the information to sketch the graph.

$\left(\begin{array}{l} f(0)=0 \\ f(x)=0 \text{ only if } x=0 \end{array} \right)$

i, $f(x)$ not defined at $x=-2$ $\quad \lim_{x\to -2^-} \frac{x}{x+2} = +\infty$

$\lim_{x\to -2^+} \frac{x}{x+2} = -\infty$ $\quad \therefore$ $\boxed{x=-2 \text{ is vertical asymptote}}$

$\lim_{x\to\infty} \frac{x}{x+2} = 1$, $\lim_{x\to -\infty} \frac{x}{x+2} = 1$ $\quad \therefore$ $\boxed{y=1 \text{ is horizontal asymptote}}$

ii, $f'(x) = \frac{(1)(x+2) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2} > 0$ (for all $x \neq -2$) so $\boxed{f(x) \text{ always increasing}}$

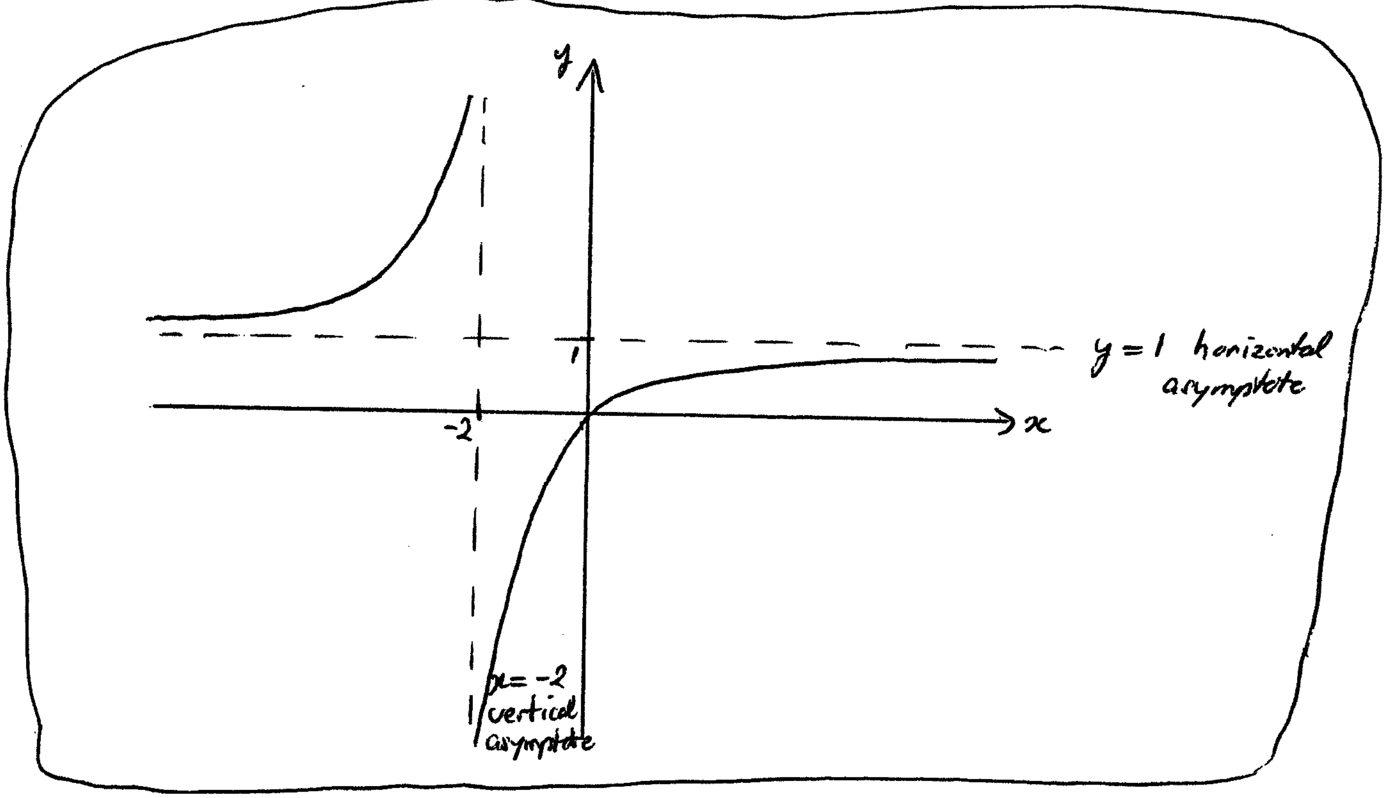
and $\boxed{\text{no local extrema}}$

iii, $f''(x) = \frac{-4}{(x+2)^3}$ $\quad$ if $\boxed{x<-2,\ f''(x)>0,\ f(x) \text{ concave up}}$

if $\boxed{x>-2,\ f''(x)<0,\ f(x) \text{ concave down}}$

but $\boxed{\text{no inflection point}}$

iv,



SOL: B

1. [3 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{\cos^2 x \arcsin x}{(x+1)^6}$.

NOTE that: $\ln(f(x)) = \ln\left[\dfrac{\cos^2 x \cdot \arcsin x}{(x+1)^6}\right] =$

$= 2\ln(\cos x) + \ln(\arcsin x) - 6\ln(x+1)$.

So: $\dfrac{d}{dx}\left(\ln(f(x))\right) = 2 \cdot \dfrac{-\sin x}{\cos x} + \dfrac{\frac{1}{\sqrt{1-x^2}}}{\arcsin x} - 6 \cdot \dfrac{1}{x+1}$

BUT $\dfrac{d}{dx}\left(\ln(f(x))\right) = \dfrac{f'(x)}{f(x)}$

So: $f'(x) = \dfrac{\cos^2 x \cdot \arcsin x}{(x+1)^6}\left[\dfrac{-2\sin x}{\cos x} + \dfrac{\frac{1}{\sqrt{1-x^2}}}{\arcsin x} - \dfrac{6}{x+1}\right]$

B

2. [3 points] A ladder 4 m long is leaning against a wall. The top of the ladder is sliding down the wall at 0.25 m/s. How fast is the bottom of the ladder sliding along the floor when the ladder is 1 m from the wall?

want $\frac{dx}{dt}$ when $x = 1$ m

when $x = 1$ m, $y = \sqrt{16-(1)^2} = \sqrt{15}$ m

so $\frac{dx}{dt} = -\frac{\sqrt{15}}{1}(-0.25\,m/s) = \boxed{\frac{\sqrt{15}}{4}\,m/s} \simeq \boxed{0.9682\,m/s}$

B

3. [2 points] Find the absolute maximum and minimum of $f(x) = x - 3\ln x$ on $[1, 5]$.

$$f'(x) = 1 - \frac{3}{x}$$

$$f'(x) = 0 \quad \text{if} \quad x = 3 \in [1, 5]$$

$$f(1) = 1 - 3\ln 1 = 1$$

$$f(3) = 3 - 3\ln 3 \simeq -0.2958$$

$$f(5) = 5 - 3\ln 5 \simeq 0.1717$$

$\therefore$ $\boxed{\text{max is } 1}$ (at $x = 1$)

and $\boxed{\text{min is } 3 - 3\ln 3}$ (at $x = 3$)

4. [4 points] Find the following limits:

(a) $\lim_{x\to 0+} \dfrac{e^x - \cos x - x}{5000x^2}$ (b) $\lim_{x\to\infty} \left(1 - \dfrac{8}{x}\right)^x$

a) $\lim_{x\to 0^+} \dfrac{e^x - \cos x - x}{5000x^2}$ IT IS $\frac{0}{0} \to$ L'H

$= \lim_{x\to 0^+} \dfrac{e^x + \sin x - 1}{2\cdot 5000 \cdot x}$ iT is $\frac{0}{0} \to$ L'H

$= \lim_{x\to 0^+} \dfrac{e^x + \cos x}{2\cdot 5000} = \dfrac{2}{2\cdot 5000} = \boxed{\dfrac{1}{5000}}$

(b) $\lim_{x\to\infty} \left(1 - \dfrac{8}{x}\right)^x$ iT is 1^∞;

Set $y = \left(1 - \dfrac{8}{x}\right)^x$; Then $\ln y = x \ln\left(1 - \dfrac{8}{x}\right)$.

NOTE: $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 - \dfrac{8}{x}\right) = \lim_{x\to\infty} \dfrac{\ln\left(1-\frac{8}{x}\right)}{x^{-1}}$ $\boxed{\frac{0}{0}}$

$\overset{L'H}{=} \lim_{x\to\infty} \dfrac{\frac{1}{1-\frac{8}{x}}\left(\frac{8}{x^2}\right)}{-x^{-2}} = \lim_{x\to\infty} \dfrac{-8}{1-\frac{8}{x}} = -8$

Hence $\ln\left[\lim_{x\to\infty} y\right] = -8$; THUS

$\lim_{x\to\infty} y = e^{-8}$ OR: $\lim_{x\to\infty} \left(1 - \dfrac{8}{x}\right)^x = e^{-8}$

Recall how we defined y !!!

5. [2 points] If $f''(x) = 6e^{3x} + 12x^2 + 4\cos x - 1$, find the most general f.

$\Downarrow$

$$f'(x) = \frac{6}{3} e^{3x} + \frac{12}{3} x^3 + 4\sin x - x + C, \quad C\#$$

$\Downarrow$

$$f(x) = \frac{6}{3\cdot 3} e^{3x} + \frac{12}{3\cdot 4} x^4 - 4\cos x - \frac{x^2}{2} + Cx + D,$$

D#

So:

$$f(x) = \frac{2}{3} e^{3x} + x^4 - 4\cos x - \frac{x^2}{2} + Cx + D$$

C, D #S

B

6. [6 points] Consider the function $y = f(x) = \dfrac{x}{x+4}$.

(i) Find any vertical or horizontal asymptotes.
(ii) Find the intervals of increase and decrease and any local extrema.
(iii) Find the intervals of concavity and any inflection points.
(iv) Use all of the information to sketch the graph.

$\left(f(0) = 0 \right.$
$\left. f(x) = 0 \text{ only if } x = 0 \right)$

i, $f(x)$ not defined at $x = -4$ $\quad \lim_{x \to -4^-} \frac{x}{x+4} = \infty$

$\lim_{x \to -4^+} \frac{x}{x+4} = -\infty$ $\quad \boxed{x = -4 \text{ vert. asympt.}}$

$\lim_{x \to \infty} \frac{x}{x+4} = 1$, $\lim_{x \to -\infty} \frac{x}{x+4} = 1$ $\quad \boxed{y = 1 \text{ horiz. asympt.}}$

ii, $f'(x) = \dfrac{4}{(x+4)^2} > 0 \quad (x \neq -4)$ $\quad \boxed{f(x) \text{ always inc.}}$ $\quad \boxed{\text{no local extrema}}$

iii, $f''(x) = \dfrac{-8}{(x+4)^3}$

$x < -4$	$f''(x) > 0$	$f(x)$ concave up
$x > -4$	$f''(x) < 0$	$f(x)$ concave down

but $\boxed{\text{no inf. pt.}}$

iv,