

DGD (circle one):

Last name: **ANSWERS**

DGD1 (in DMS1120)

First name:

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Student number:

Marks: /10

MAT 1348B — Fourth Homework Assignment (Prof. P. Scott) — ANSWERS
Due Feb. 11, 2015 by 3:30pm

1. Give an **indirect proof** (i.e. by **contraposition**) of the following theorem:

Let m and n be integers. If mn is even, then m is even or n is even.

[5pts]

SOLUTION. We prove the contrapositive. That is, informally, if $\neg(m \text{ even or } n \text{ even})$ then $\neg(mn \text{ even})$. Translating into propositional logic and using de Morgan's law, let P be m is even, Q be n is even, R be mn is even. We want the contrapositive $(\neg(P \vee Q) \rightarrow \neg R) \equiv ((\neg P \wedge \neg Q) \rightarrow \neg R)$. This says

$$m \text{ odd and } n \text{ odd} \rightarrow mn \text{ odd.}$$

This is easy to prove: suppose m and n are odd. Then $m = 2k + 1$ and $n = 2\ell + 1$ for some integers k, ℓ . Then

$$mn = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1, \text{ which is odd.}$$

2. An integer n is called a **perfect square** if $n = m^2$ for some integer m .

Give a **proof by contradiction** of the following theorem:

Let n be an integer. If n is a perfect square, then $n + 2$ is not a perfect square.

[5pts]

SOLUTION. Suppose n is a perfect square, but suppose (for contradiction) that $n + 2$ is also a perfect square. Then $n = k^2$ and $n + 2 = s^2$, for some integers k, s . Then we see:

$$2 = (n + 2) - n = s^2 - k^2 = (s + k)(s - k) \tag{1}$$

Since 2 is a prime number, the only *positive* divisors are 2 and 1. So, either

$$s + k = 2 \quad \text{and} \quad s - k = 1, \text{ or} \tag{2}$$

$$s + k = 1 \quad \text{and} \quad s - k = 2 \tag{3}$$

For (2): suppose $s + k = 2$ and $s - k = 1$. Then $2 = s + k = (1 + k) + k = 1 + 2k$, so $k = 1/2$: contradiction, since k is an integer.

For (3): suppose $s + k = 1$ and $s - k = 2$. Then $1 = s + k = (2 + k) + k = 2 + 2k$, so $k = -1/2$, a contradiction, since k is an integer.

So in either case, we get a contradiction.

Remark: The above suffices for positive divisors of 2. In fact, if $n > 0$ and we assume $k, s > 0$; then since $n + 2 > n$, we see $s + k > s - k$, so case (2) suffices (i.e. we definitely know $s + k = 2$ and $s - k = 1$.) If $n = 0$, then notice 2 is not a perfect square (indeed, we actually showed in class $\sqrt{2}$ is not even rational!). I leave the case of divisors $-1, -2$ to you.