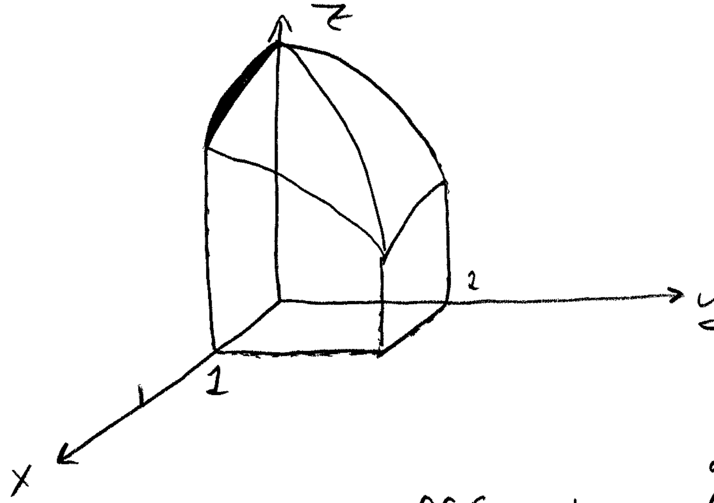
1. Consider the solid in the first octant bounded by the planes $z=0$, $x=0$, $y=0$, $x=1$, $y=2$ and the paraboloid $z=9-x^2-y^2$. This solid has a mass density given by the function $\delta(x,y,z)=xy$. Find the total mass of this solid.



$$\text{Mass} = \iiint_V \delta\, dV = \int_0^2 \int_0^1 \int_0^{9-x^2-y^2} xy \, dz\, dx\, dy$$

$$\left(\text{or } \int_0^1 \int_0^2 \int_0^{9-x^2-y^2} xy \, dz\, dy\, dx \right)$$

$$= \int_0^2 \int_0^1 \left(xyz\right)\Big|_{z=0}^{z=9-x^2-y^2} dx\, dy = \int_0^2 \int_0^1 xy(9-x^2-y^2)\, dx\, dy =$$

$$\int_0^2 \int_0^1 (9xy - x^3y - xy^3)\, dx\, dy = \int_0^2 \left(\frac{9}{2}x^2y - \frac{x^4}{4}y - \frac{x^2y^3}{2} \Big|_0^1 \right) dy = \int_0^2 \left(\left(\frac{9}{2}-\frac{1}{4}\right)y - \frac{1}{2}y^3 \right) dy$$

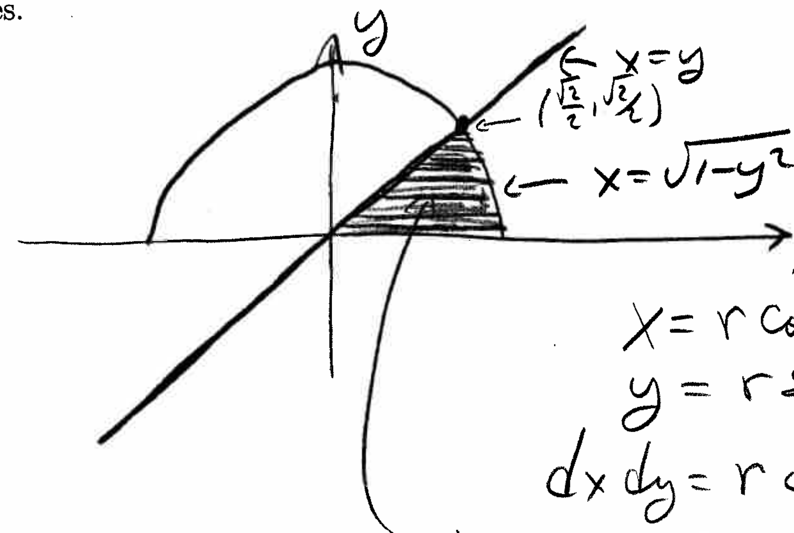
$$= \int_0^2 \left(\frac{17}{4}y - \frac{1}{2}y^3 \right) dy = \frac{17}{8}y^2 - \frac{1}{8}y^4 \Big|_0^2 = \frac{17 \cdot 4}{8} - 16/8 = \frac{17}{2} - \frac{4}{2} = 13/2$$

$$= 6.5$$

2. Convert the following integral into polar coordinates. **DO NOT EVALUATE THE INTEGRAL.**

$$\int_0^{\frac{\sqrt{2}}{2}} \int_y^{\sqrt{1-y^2}} (x+y)\,dx\,dy$$

Hint: Sketch the region of integration in the x-y plane, and then express this region in terms of polar coordinates.



$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta = r\,d\theta\,dr$

this region is $0 \le r \le 1$
$0 \le \theta \le \pi/4$

$$\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} (x+y)\,dx\,dy = \int_0^{\pi/4} \int_0^1 (\underbrace{r\cos\theta}_{x} + \underbrace{r\sin\theta}_{y})\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^1 r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$

$$\left(\text{or } \int_0^1 \int_0^{\pi/4} r^2(\cos\theta + \sin\theta)\,d\theta\,dr\right)$$

Approach 1 : Spherical coords.

3. Consider the solid drawn on the following page which is bounded by the hemisphere $z = \sqrt{1-x^2-y^2}$, the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 0$. This solid has a mass density given by the function $\delta(x, y, z) = x^2 + y^2$. Using either spherical coordinates or cylindrical coordinates (your choice), set up a triple integral which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

The solid is defined by the inequalities

$0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad \pi/4 \le \phi \le \pi/2$

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

$$\text{Mass} = \iiint_V \delta \, dV$$

$$= \int_{\phi=\pi/4}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \Big(\underbrace{\rho^2 \cos^2\theta \sin^2\phi}_{x^2} + \underbrace{\rho^2 \sin^2\theta \sin^2\phi}_{y^2}\Big) \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_{\phi=\pi/4}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \rho^4 \sin^3\phi \, d\rho\, d\theta\, d\phi$$

Spherical coords

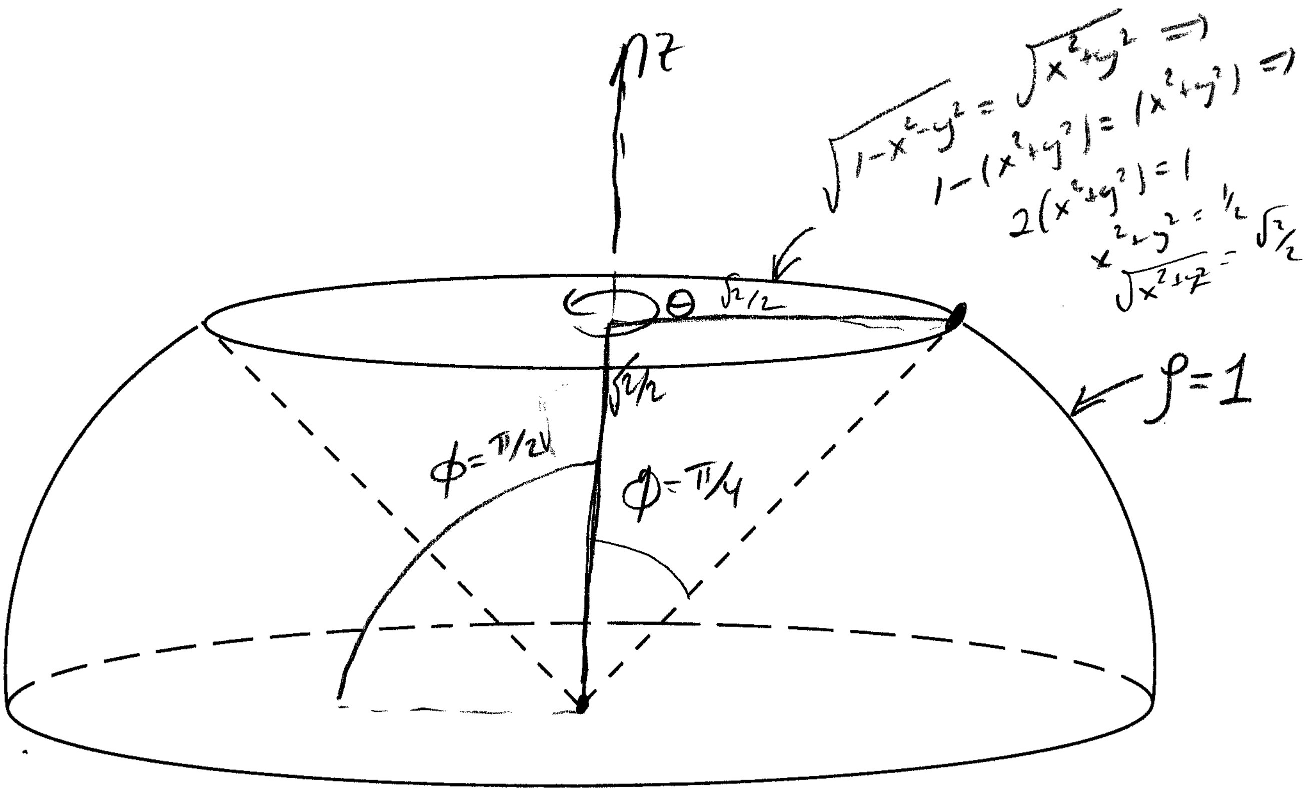


Figure 1: Figure representing the solid region described in problem 3.

Approach 2: Cylindrical coords

3. Consider the solid drawn on the following page which is bounded by the hemisphere $z = \sqrt{1-x^2-y^2}$, the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 0$. This solid has a mass density given by the function $\delta(x,y,z) = x^2+y^2$. Using either spherical coordinates or cylindrical coordinates (your choice), set up a triple integral which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

The solid is defined by the inequalities

$z \le r \le \sqrt{1-z^2}$, $0 \le \theta \le 2\pi$, $0 \le z \le \sqrt{2}/2$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\text{Mass} = \iiint_V \delta\, dV$$

$$= \int_0^{\sqrt{2}/2}\int_0^{2\pi}\int_z^{\sqrt{1-z^2}} (\underbrace{r^2\cos^2\theta}_{x^2} + \underbrace{r^2\sin^2\theta}_{y^2})\, r\,dr\,d\theta\,dz$$

$$= \int_0^{\sqrt{2}/2}\int_0^{2\pi}\int_z^{\sqrt{1-z^2}} r^3\,dr\,d\theta\,dz$$

Cylindrical coords

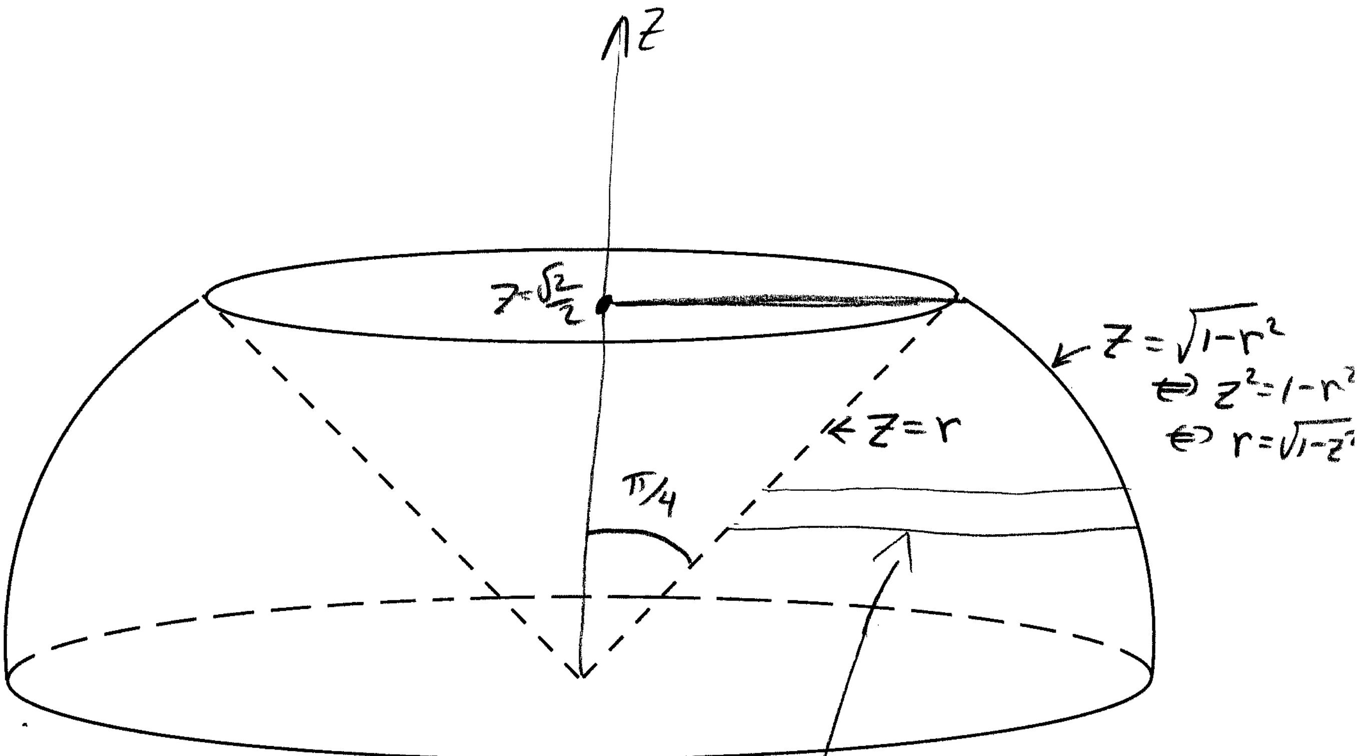


Figure 1: Figure representing the solid region described in problem 3.

$$z \le r \le \sqrt{1-z^2}$$

$$0 \le z \le \sqrt{2}/2$$

$$0 \le \theta \le 2\pi$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \underbrace{\cos t}_{x(t)}\,\vec{\mathbf{i}} + \underbrace{t}_{y(t)}\,\vec{\mathbf{j}} + \underbrace{\sin t}_{z(t)}\,\vec{\mathbf{k}}, \quad 0 \le t \le 4\pi.$$

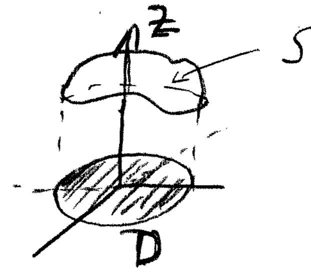
$$\text{Length} = \int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^{4\pi} \sqrt{(-\sin t)^2 + 1^2 + (\cos t)^2}\, dt$$

$$= \int_0^{4\pi} \sqrt{1 + \cos^2 t + \sin^2 t}\, dt = \int_0^{4\pi} \sqrt{2}\, dt = \sqrt{2}\, t \Big/_0^{4\pi}$$

$$= 4\sqrt{2}\,\pi$$

Approach 1
Polar coordinates for D



5. Let D be the unit disk in the x-y plane, i.e.

$$D = \{ (x, y) \,|\, 0 \leq x^2 + y^2 \leq 1 \}$$

and let S be the graph of the function $z = 4 + y^2 - x^2$ defined for points (x, y) in D.

(a) Give a parametrization of this surface S.

(b) Set up an integral which would give the total area of this surface S, **but do not evaluate this integral**.

(c) **BONUS [2 marks]** Evaluate the integral in (b). Note that you are eligible to receive bonus marks **only if you have the correct answer in (b).**

(a) Using polar coordinates for D, we write

$$\vec{r}(a,\theta) = \underbrace{a\cos\theta}_{x(a,\theta)}\,\vec{\imath} + \underbrace{a\sin\theta}_{y(a,\theta)}\,\vec{\jmath} + \underbrace{(4 + a^2\sin^2\theta - a^2\cos^2\theta)}_{z(a,\theta) = 4+y^2-x^2}\vec{k}$$

$0 \leq a \leq 1$

$0 \leq \theta \leq 2\pi$

(b)
$$\vec{r}_a \times \vec{r}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\theta & \sin\theta & 2a(\sin^2\theta - \cos^2\theta) \\ -a\sin\theta & a\cos\theta & \underbrace{2a^2\sin\theta\cos\theta + 2a^2\sin\theta\cos\theta}_{4a^2\sin\theta\cos\theta} \end{vmatrix}$$

$$= \vec{\imath}\left(4a^2\sin^2\theta\cos\theta - 2a^2\sin^2\theta\cos\theta + 2a^2\cos^3\theta\right)$$

$$- \vec{\jmath}\left(4a^2\sin\theta\cos^2\theta + 2a^2\sin^3\theta - 2a^2\sin\theta\cos^2\theta\right)$$

$$+ \vec{k}\left(a\cos^2\theta + a\sin^2\theta\right)$$

(Extra page)

$$= \vec{i}\left(2a^2\cos\theta\,(\sin^2\theta+\cos^2\theta)\right)$$
$$-\vec{j}\left(2a^2\sin\theta\,(\sin^2\theta+\cos^2\theta)\right)+a\vec{k}$$

$$= 2a^2\cos\theta\,\vec{i} - 2a^2\sin\theta\,\vec{j} + a\vec{k}$$

$$\|\vec{r}_a\times\vec{r}_\theta\| = \sqrt{\underbrace{4a^4\cos^2\theta+4a^4\sin^2\theta}_{4a^4}+a^2} = \sqrt{a^2(4a^2+1)}$$
$$= a\sqrt{1+4a^2}$$

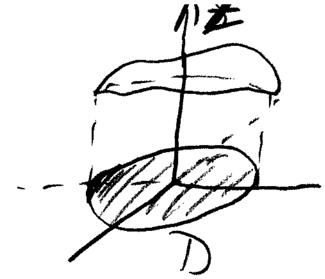
Therefore $\text{Area} = \iint_D \|\vec{r}_a\times\vec{r}_\theta\|\,da\,d\theta = \int_0^{2\pi}\int_0^1 a\sqrt{1+4a^2}\,da\,d\theta$

(c) By using a substitution, we get that an antiderivative for $a\sqrt{1+4a^2}$ is $\frac{1}{12}(1+4a^2)^{3/2}$ so

$$\text{Area} = \int_0^{2\pi}\left(\frac{1}{12}(1+4a^2)^{3/2}\Big|_0^1\right)d\theta = \int_0^{2\pi}\frac{1}{12}\left(5^{3/2}-1^{3/2}\right)d\theta$$

$$= \frac{2\pi}{12}\left(5^{3/2}-1\right) = \frac{\pi}{6}\left(5^{3/2}-1\right)$$

Approach 2
Rectangular coordinates for D



5. Let D be the unit disk in the x-y plane, i.e.

$$D = \{\,(x,y)\,|\,0 \le x^2 + y^2 \le 1\,\}$$

and let S be the graph of the function $z = 4 + y^2 - x^2$ defined for points (x,y) in D.

(a) Give a parametrization of this surface S.

(b) Set up an integral which would give the total area of this surface S, **but do not evaluate this integral**.

(c) **BONUS [2 marks]** Evaluate the integral in (b). Note that you are eligible to receive bonus marks **only if you have the correct answer in (b)**.

a) Using rectangular coordinates for D, we write

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (4+y^2-x^2)\vec{k}$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$-1 \le x \le 1$$

b) $$\vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & 2y \end{vmatrix} =$$

$$\vec{i}\,2x - \vec{j}\,2y + \vec{k} = 2x\vec{i} - 2y\vec{j} + \vec{k}$$

$$\|\vec{r_x} \times \vec{r_y}\| = \sqrt{4x^2 + 4y^2 + 1}.$$ Therefore

$$\text{Area} = \iint_D \sqrt{4x^2+4y^2+1}\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2+4y^2+1}\,dy\,dx$$

(Extra page)

c) In order to be able to evaluate the integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2+4y^2+1}\, dy\, dx$, we will convert to polar coordinates: D is described by

$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$x = r\cos\theta$
$y = r\sin\theta$
$dy\,dx = r\,dr\,d\theta$

So $\text{Area} = \int_0^{2\pi}\int_0^1 \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\; r\,dr\,d\theta$

see previous approach

$$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\; r\,dr\,d\theta = \ldots$$

$$= \frac{\pi}{6}\left(5^{3/2}-1\right)$$