MATH 1005A
Test 1 Solutions

February 3, 2015

[Marks]

[6] 1. Solve the initial-value problem $y' = \dfrac{3x^2}{4y^3}$, $y(0) = 2$.

Solution:

The equation is separable, $4y^3y' = 3x^2 \;\Rightarrow\; y^4 = x^3 + c \;\Rightarrow\; y = (x^3+c)^{1/4}$. $y(0) = 2$ $\Rightarrow\; c^{1/4} = 2 \;\Rightarrow\; c = 16 \;\Rightarrow\; y = (x^3+16)^{1/4}$.

[6] 2. Find the general solution of $y' = \dfrac{x^2+2y^2}{2xy}$.

Solution:

The equation is homogeneous, $y' = \dfrac{x^2+2y^2}{2xy} = \dfrac{x}{2y} + \dfrac{y}{x}$, $u = \dfrac{y}{x} \;\Rightarrow\; y = xu \;\Rightarrow$ $y' = u + xu'$, and the equation becomes $u + xu' = \dfrac{1}{2u} + u$, or $xu' = \dfrac{1}{2u}$, which is separable. Then $2uu' = \dfrac{1}{x} \;\Rightarrow\; u^2 = \ln|x| + c \;\Rightarrow\; u = \pm\sqrt{\ln|x|+c} \;\Rightarrow$ $y = \pm x\sqrt{\ln|x|+c}$.

[6] 3. Find the general solution of $xy' + 3y = 4x - \dfrac{1}{x^3}$, $x > 0$.

Solution:

The equation is linear, with the standard form $y' + \dfrac{3}{x}y = 4 - \dfrac{1}{x^4}$, and the integrating factor $I(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln(x)} = x^3$. The equation then becomes $x^3y' + 3x^2y = 4x^3 - \dfrac{1}{x}$, i.e., $(x^3y)' = 4x^3 - \dfrac{1}{x}$. Hence, $x^3y = x^4 - \ln(x) + c$, and $y = x - \dfrac{\ln(x)}{x^3} + \dfrac{c}{x^3}$.

[6] 4. Find the general solution of $2y - 3x^2 + (2x - 3y^2)y' = 0$.

Solution:

The equation is exact since $P_y = 2 = Q_x$. $f_x = P = 2y - 3x^2 \;\Rightarrow$ $f(x,y) = 2xy - x^3 + g(y)$, and $f_y = Q \;\Rightarrow\; 2x + g'(y) = 2x - 3y^2 \;\Rightarrow\; g(y) = -y^3 + c$ $\Rightarrow\; f(x,y) = 2xy - x^3 - y^3 + c$, and the general solution is $2xy - x^3 - y^3 = k$.

[6] 5. Find the general solution of $3x^3y^2 + 1 + 2x^4yy' = 0$.

Solution:

The equation is not exact since $P_y = 6x^3y$, $Q_x = 8x^3y$, and $P_y \neq Q_x$. Since $\dfrac{P_y - Q_x}{Q} = \dfrac{-2x^3y}{2x^4y} = -\dfrac{1}{x}$ is independent of y, an integrating factor $I(x)$ exists and is given by $\dfrac{I'(x)}{I(x)} = -\dfrac{1}{x}$. Hence, $\ln|I(x)| = -\ln|x| = \ln(|x|^{-1}) \;\Rightarrow\; I(x) = \pm\dfrac{1}{x}$. With

$I(x) = \frac{1}{x}$, the equation becomes $3x^2y^2 + \frac{1}{x} + 2x^3yy' = 0$, and is exact.
$f_x = P = 3x^2y^2 + \frac{1}{x} \;\Rightarrow\; f(x, y) = x^3y^2 + \ln|x| + g(y)$, and $f_y = Q \;\Rightarrow$
$2x^3y + g'(y) = 2x^3y \;\Rightarrow\; g(y) = c \;\Rightarrow\; f(x, y) = x^3y^2 + \ln|x| + c$, and the general solution is $x^3y^2 + \ln|x| = k$.