Physics 304 **Midterm** **February 28, 2014**

Note: (1) There are 2 questions; each of them is worth 15 points.
(2) The use of the textbook (Griffiths) or formula sheet is permitted.
(3) The use of a calculator is permitted.

(1) Consider, quantum mechanically, a stream of non-interacting particles of mass *m* moving along the negative *x* direction (from right to left) toward a potential step located at $x = 0$. The potential is zero for $x \geq 0$, and $V_0 > 0$ for $x < 0$.

(a) Starting from the Schrödinger equation and imposing appropriate boundary conditions on the wave functions, for the case $E = p^2/2m > V_0$ calculate the fraction of reflected and transmitted particles at $x = 0$, verifying that they add up to 100%.

(b) What would transmission and reflection coefficients be if $E = p^2/2m < V_0$? In this particular case, support your conclusions with an explicit calculation based on the probability currents.

(c) And what would transmission and reflection coefficients be, if $E = p^2/2m = V_0$?

(2) A particle in an infinite square well of width *a* has an initial wave function

$$\Psi(x,0) = A\left(\sqrt{2}\psi_1(x) + \sqrt{5}\psi_2(x)\right)$$

where $\psi_n(x)$ are the mutually orthogonal, normalized stationary states.

(a) What is the normalization constant A?

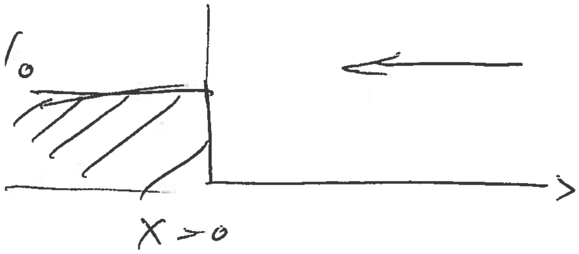
(b) Write the time-dependent wave function $\Psi(x,t)$, as well as $|\Psi(x,t)|^2$.

(c) If the energy were measured, what would be the possible outcomes? And their probabilities?

(d) What is the expectation value of the total energy of this state? Is it time dependent?

(e) Write an expression for the probability of finding the particle in the right-half of the well as a function of time. Look carefully at the integrals you obtain (*some of them might be trivially solved*), and establish which ones vanish and which ones do not (*you do not have to complete the computations of non-vanishing integrals all the way*). Is this probability time dependent?

Problem # 1



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

a) $\boxed{x < 0}$ $\qquad$ $\boxed{x > 0}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \qquad k_L = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V_0)\psi = -k_L^2\psi$$

$$\Rightarrow \underline{\psi_L = T e^{-ik_L x}}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k_R^2\psi \qquad k_R = \sqrt{\frac{2m}{\hbar^2}}$$

$$\Rightarrow \underline{\psi_R = e^{-ik_R x} + R e^{ik_R x}}$$

Boundary conditions
$$\begin{cases} \psi_L(0) = \psi_R(0) \Rightarrow T = 1 + R \\ \psi_L'(0) = \psi_R'(0) \Rightarrow -ik_L T = -ik_R + R\,ik_R \end{cases}$$

$$\Rightarrow \underline{R = \frac{k_R - k_L}{k_R + k_L}}, \qquad \underline{T = \frac{2k_R}{k_R + k_L}}$$

$$\Rightarrow \left.\begin{aligned} \mathbb{R} &= \frac{J_R}{J_I} = \frac{\hbar k_R}{m}|R|^2 \Big/ \frac{\hbar k_R}{m} = |R|^2 = \left(\frac{k_R - k_L}{k_R + k_L}\right)^2 \\ \mathbb{T} &= \frac{J_T}{J_I} = \frac{\hbar k_L}{m}|T|^2 \Big/ \frac{\hbar k_R}{m} = \frac{k_L}{k_R}|T|^2 = \frac{4k_L k_R}{(k_L + k_L)^2} \end{aligned}\right\} \underline{\mathbb{R} + \mathbb{T} = 1}$$

b) For $E < V_0$, the particle is reflected back. $\mathbb{R} = 100\%$

How can I show it?

there will be some penetration in the barrier.

$$K_L = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} = i\,|K_L| \quad \text{because } (E-V_0) < 0$$

$$\Rightarrow \psi_L = T e^{-ik_L x} = \underbrace{T e^{+|K_L| x}}_{} \quad \text{for } \underline{x<0}$$

$\hookrightarrow$ decaying exponential $\in \mathbb{R}$

for $\mathbb{T} = \frac{J_T}{J_I}$ an $J_T = \frac{\hbar}{2im}\left[\psi_T^* \frac{\partial \psi_T}{\partial x} - \frac{\partial \psi_T^*}{\partial x}\psi_T\right]$

but because $\psi_T \in \mathbb{R} \Rightarrow$ $\left.\begin{array}{l}\psi_T^* = \psi_T \\ \frac{\partial \psi_T^*}{\partial x} = \frac{\partial \psi_T}{\partial x}\end{array}\right\} \Rightarrow \underline{J_T = 0}$

$$\Rightarrow \boxed{\mathbb{T} = 0 \quad \text{and} \quad \mathbb{R} = 1} \quad \text{because } \underline{\mathbb{T} + \mathbb{R} = 1}$$

c) if $E = V_0 \Rightarrow K_L = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} = 0$!

$$\frac{d^2\psi}{dx^2} = 0 \Rightarrow \underline{\psi_L(x) = Ax + B} \in \mathbb{R} \,!!$$

$\left\{\begin{array}{lll}\psi \text{ continuous at } x=0 & B = 1+R \\ \psi' \quad \text{"} \quad \text{at } x=0 & A = -ik_R + ik_R R\end{array}\right\}$

the key point is that ψ_L is again $\in \mathbb{R}$

$\Rightarrow \boxed{\mathbb{T} = 0 \,;\, \mathbb{R} = 1}$

if we don't like linear increase in x, we can impose $A = 0$ $\Bigg\}$ $\begin{array}{l}R = 1 \\ B = 2\end{array}$

②

Problem #2

$\psi(x,0) = A\left[\sqrt{2}\,\psi_1(x) + \sqrt{5}\,\psi_2(x)\right]$ with ψ_1, ψ_2 eigenstate of infinite square well.

a) Calculate A.

$$\int_0^a |\psi(x,0)|^2 dx = |A|^2 \int_0^a (2\psi_1^2 + 5\psi_2^2)\,dx = |A|^2(2+5) = 7|A|^2 = 1$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{7}}}$$ where we exploited $\int_0^a \psi_{1,2}^2\,dx = 1$

b) $$\psi(x,t) = \frac{1}{\sqrt{7}}\left[\sqrt{2}\,\psi_1(x)\,e^{-iE_1t/\hbar} + \sqrt{5}\,\psi_2(x)\,e^{-iE_2t/\hbar}\right]$$

$$|\psi(x,t)|^2 = \psi^*(x,t)\,\psi(x,t) = \frac{1}{7}\left[\sqrt{2}\,\psi_1\,e^{iE_1t/\hbar} + \sqrt{5}\,e^{iE_2t/\hbar}\right]\left[\sqrt{2}\,\psi_1\,e^{-iE_1t/\hbar} + \sqrt{5}\,\psi_2\,e^{-iE_2t/\hbar}\right]$$

$$= \frac{1}{7}\left[2\psi_1^2 + 5\psi_2^2 + \sqrt{10}\,\psi_1\psi_2\left(e^{-i(E_2-E_1)t/\hbar} + e^{+i(E_2-E_1)t/\hbar}\right)\right] =$$

$$= \frac{1}{7}\left[2\psi_1^2 + 5\psi_2^2 + 2\sqrt{10}\,\psi_1\psi_2\cos\frac{(E_2-E_1)t}{\hbar}\right]$$

c) Measurement of the energy would give

E_1 with probability $\left(\frac{\sqrt{2}}{\sqrt{7}}\right)^2 = \frac{2}{7}$

E_2 with probability $\left(\frac{\sqrt{5}}{\sqrt{7}}\right)^2 = \frac{5}{7}$

Probability add to 100% !

d) $$\langle \hat{H} \rangle = E_1P_1 + E_2P_2 = E_2\frac{2}{7} + E_2\frac{5}{7} = \frac{\hbar^2\pi^2}{2ma^2}\left(1\cdot\frac{2}{7} + 4\cdot\frac{5}{7}\right) = \frac{22}{7}\cdot\frac{\hbar^2\pi^2}{2ma^2}$$

Because of the orthonormality of $\psi_1(x,t)$ and $\psi_2(x,t)$, which is independent of time (orthonormality is defined as an integral in dx), it follows that:

$$\langle \hat{H} \rangle_{\psi(x,t)} = \langle \psi(x,t) | \hat{H} | \psi(x,t) \rangle = \frac{2}{7} E_1 \int |\psi_1(x,t)|^2 dx + \frac{5}{7} E_2 \int |\psi_2(x,t)| dx =$$

$$= \frac{2}{7} E_1 + \frac{5}{7} E_2 = \langle \hat{H} \rangle_{\psi(x,0)} \quad \underline{\text{TIME INDEPENDENT}}$$

e) $$P\left(\frac{a}{2} < x < a\right) = \int_{a/2}^{a} dx \, |\psi(x,t)|^2 =$$

$$\frac{1}{7} \left\{ 2 \underbrace{\int_{a/2}^{a} \psi_1^2 \, dx}_{1/2} + 5 \underbrace{\int_{a/2}^{a} \psi_2^2 \, dx}_{1/2} + 2\sqrt{10} \cos \frac{(E_2 - E_1)t}{\hbar} \int_{a/2}^{a} \psi_1 \psi_2 \, dx \right\}$$

$$= \frac{1}{7} \left(\frac{2+5}{2} \right) + \frac{2}{7} \sqrt{10} \cos \frac{(E_2 - E_1)t}{\hbar} \underbrace{\int_{a/2}^{a} \psi_1 \psi_2 \, dx}_{\neq 0} \Rightarrow \underline{\text{TIME DEPENDENT}}$$

$\int_0^a \psi_1 \psi_2 \, dx = 0$ only when integrated from 0 to a (FULL WELL!)

When integrated on 1/2 well, $\int_{a/2}^{a} \psi_1 \psi_2 \, dx \neq 0$