Problem 1

Let the impulse response of a system be given by $h(t)$ shown in Figure 1 and let $x(t)$ shown in this figure be the input excitation to this system. Use convolution to obtain expressions for the output $y(t)$.

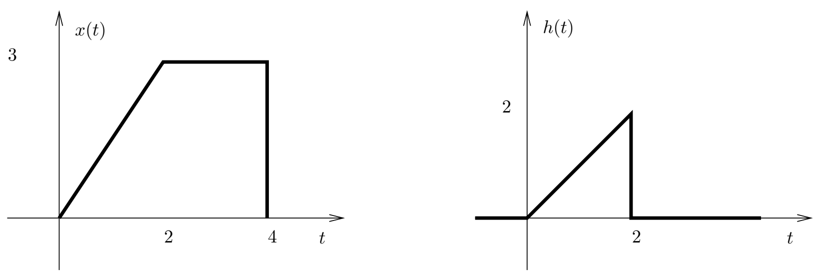

Figure 1: Input excitation and system

Solution of Problem 1

1. The output $y(t)$ is given by
$$y(t) = \int_0^t h(\tau)x(t-\tau)d\tau. \tag{1}$$
We need to compute this integration for every $t \in (-\infty, \infty)$.

2. Provide mathematical expressions for $x(t)$ and $h(t)$.
   - For $x(t)$, the slope of the line is 3/2 and it passes through the origin. Hence,
$$x(t) = \begin{cases} \frac{3}{2}t & 0 < t \leq 2, \\ 3 & 2 < t \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$
   - For $h(t)$, the slope of the line 1 and it passes through the origin. Hence,
$$x(t) = \begin{cases} t & 0 < t \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

3. To compute (1), we will flip and shift $x(\tau)$ for various intervals of t.

4. First interval: $-\infty < t < 0$. This situation is depicted in Figure 2 Since there is no overlap between $x(t-\tau)$ and $h(\tau)$, $x(t-\tau)h(\tau) = 0$ and $y(t) = 0$, for $t < 0$.

5. Second interval: $0 \leq t < 2$. This situation is depicted in Figure 3 In this region, $y(t)$ is given by
$$y(t) = \int_0^t \Big(\frac{3}{2}(t-\tau)\tau d\tau = \frac{1}{4}t^3.$$

6. Third interval: $2 \leq t < 4$. This situation is depicted in Figure 4. Notice there is a break point at $\tau = t-2$. In this region, $y(t)$ is given by
$$y(t) = \int_0^{t-2} (3)(\tau)d\tau + \int_{t-2}^{2} \Big(\frac{3}{2}(t-\tau)\tau d\tau = -\frac{1}{4}(t-2)^3 + 3t - 4.$$

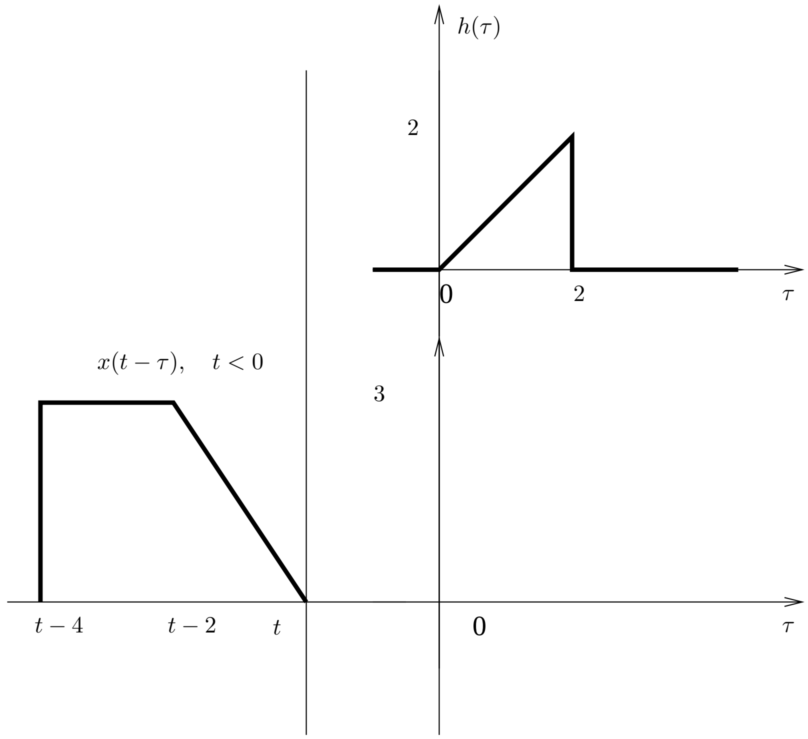

Figure 2: At $t < 0$, no overlap between $x(t-\tau)$ and $h(\tau)$.

7. Fourth interval: $4 \le t < 6$. This situation is depicted in Figure 5. In this region, $y(t)$ is given by

$$y(t) = \int_{t-4}^{2} (3)(\tau) d\tau = 6 - \frac{3}{2}(t-4)^2.$$

8. Fifth interval: $6 \le t < \infty$. For this region, there is no overlap between $x(t-\tau)$ and $h(\tau)$, and $y(t) = 0$.

9. Collecting facts:

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{4}t^3 & 0 \le t < 2 \\ -\frac{1}{4}(t-2)^3 + 3t - 4 & 2 \le t < 4 \\ 6 - \frac{3}{2}(t-4)^2 & 4 \le t < 6 \\ 0 & 6 \le t. \end{cases} \tag{2}$$

10. Checks (recommended but not necessary):
    - Check that $y(0)$ computed from the second interval is equal to $y(0)$ computed from the first interval: both are 0.
    - Check that $y(2)$ computed from the third interval is equal to $y(2)$ computed from the second interval: both are 2.
    - Check that $y(4)$ computed from the fourth interval is equal to $y(4)$ computed from the third interval: both are 6.
    - Check that $y(6)$ computed from the fourth interval is equal to $y(6)$ computed from the fifth interval: both are 0.

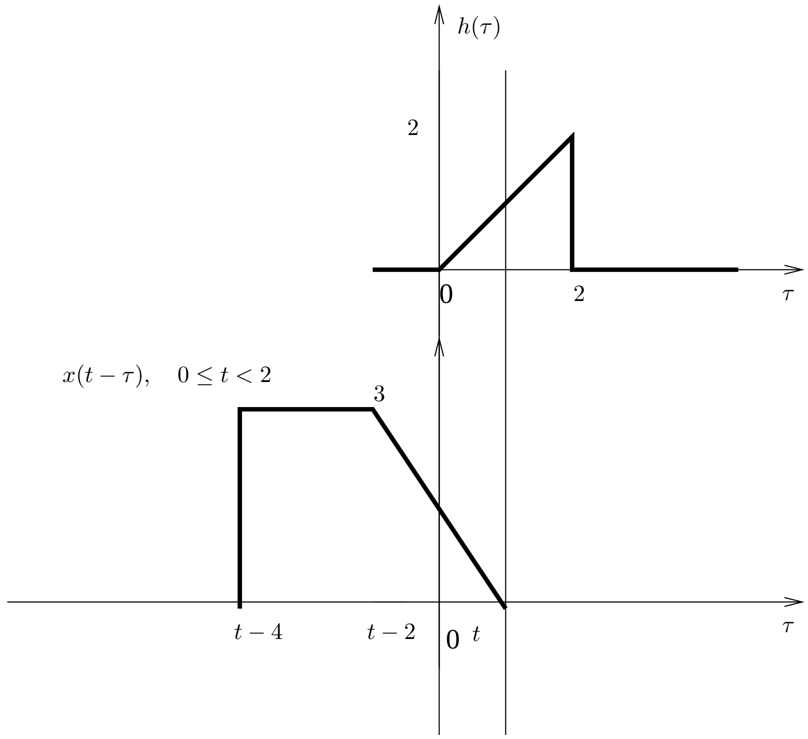

Figure 3: For $0 \le t < 2$, the overlap between $x(t-\tau)$ and $h(\tau)$ is between 0 and t.

Problem 2

Use the Laplace transform method, to obtain the outplut in the previous problem.

Solution of Problem 2

1. Express

$$x(t) = \frac{3}{2}t\big(u(t) - u(t-2)\big) + 3\big(u(t-2) - u(t-4)\big), \qquad \text{and} \tag{3}$$

$$h(t) = t\big(u(t) - u(t-2)\big). \tag{4}$$

2. To compute $X(s)$, it is easier to write expressions of the form $f(t-\alpha)u(t-\alpha)$. Hence, we can express $x(t)$ as:

$$\begin{aligned} x(t) &= \frac{3}{2}tu(t) - \frac{3}{2}(t-2+2)u(t-2) + 3\big(u(t-2) - u(t-4)\big) \\ &= \frac{3}{2}tu(t) - \frac{3}{2}(t-2)u(t-2) - 3u(t-2) + 3u(t-2) - 3u(t-4) \\ &= \frac{3}{2}tu(t) - \frac{3}{2}(t-2)u(t-2) - 3u(t-4), \end{aligned}$$

whence,

$$X(s) = \frac{3}{2s^2}(1 - e^{-2s}) - \frac{3}{s}e^{-4s}. \tag{5}$$

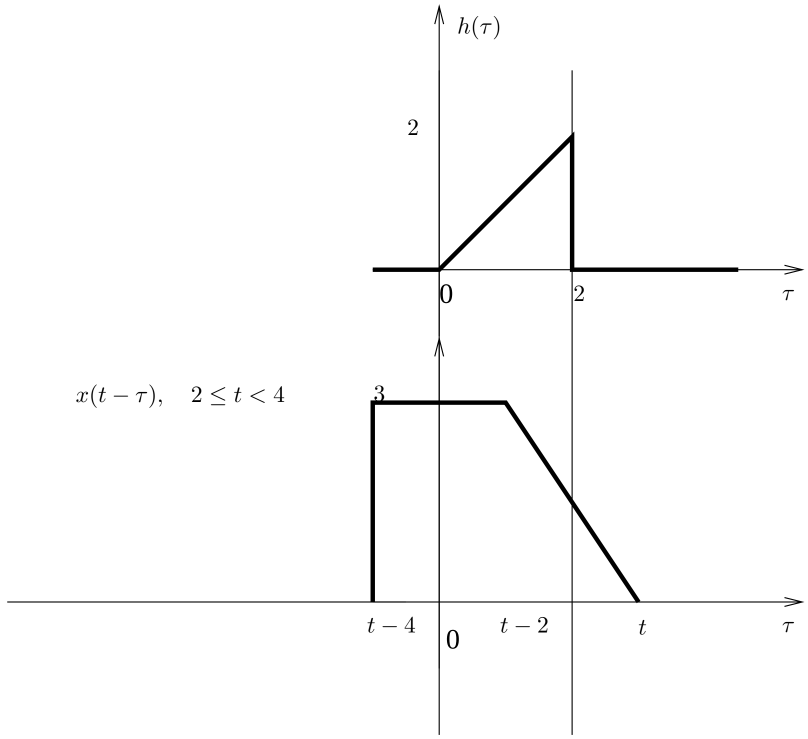

Figure 4: For $2 \leq t < 4$, the overlap between $x(t-\tau)$ and $h(\tau)$ is between 0 and 2.

3. To compute $H(s)$, we write

$$h(t) = tu(t) - (t-2+2)u(t-2) = tu(t) - (t-2)u(t-2) - 2u(t-2), \tag{6}$$

whence

$$H(s) = \frac{1}{s^2}(1 - e^{-2s}) - \frac{2e^{-2s}}{s}. \tag{7}$$

4. Now, $Y(s) = X(s)H(s)$. After straightforward simplification,

$$Y(s) = \frac{3}{2s^4}\left(1 - 2e^{-2s} + e^{-4s}\right) + \frac{6}{s^2}e^{-6s} - \frac{3}{s^3}\left(e^{-2s} - e^{-6s}\right).$$

Hence,

$$y(t) = \frac{1}{4}t^3u(t) - \frac{1}{2}(t-2)^3u(t-2) + \frac{1}{4}(t-4)^3u(t-4) + 6(t-6)u(t-6) + \frac{3}{2}(t-6)^2u(t-6) - \frac{3}{2}(t-2)^2u(t-2). \tag{8}$$

5. You can check for yourself that this solution coincides with the one obtained using convolution directly. You can try particular values of t and recall that $u(x) = 0$ for $x < 0$.

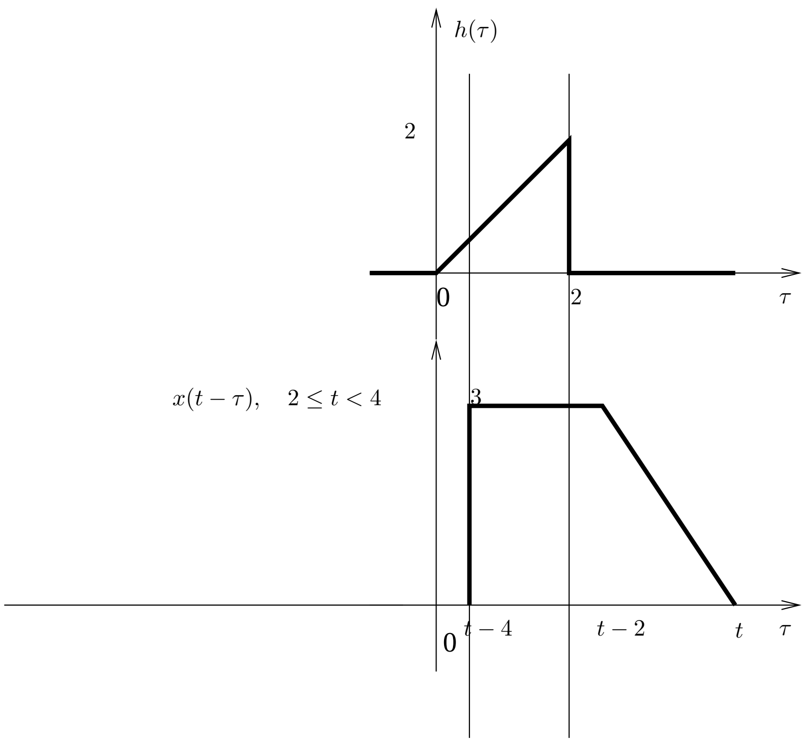

Figure 5: For $4 \leq t < 6$, the overlap between $x(t-\tau)$ and $h(\tau)$ is between $t-4$ and 2.