MAT 2355, Fall 2014
Assignment 5-Solution

(10 points)

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Question 1– [3 points] Show that an affine transformation T is an isometry if and only if its linear part is an orthogonal matrix.

Solution: Let T be an affine transformation with linear part A and translation part b. Therefore for all $x \in \mathbb{R}^n$ we have

$$T(x) = Ax + b. \tag{1}$$

- **a)** Let A be an orthogonal matrix. Then we will show that T is an isometry. We have the following theorem (Theorem 4, Week 7)

  Theorem 1. *Let A be a $n \times n$ orthogonal matrix. Then the function $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_A(v) = Av$ is an isometry.*

  But T in Equation (1) can be written as $T = T_b \circ T_A$ where $T_b(x) = x + b$. Notice that T_b, which is the translation map, is an isometry and by above theorem T_A is an isometry. Therefore $T = T_b \circ T_A$ is an isometry.

- **b)** Now assume that T is an isometry. Then we show that A is an orthogonal matrix. Notice again that $T = T_b \circ T_A$. Therefore

  $$T_{-b} \circ T = T_A.$$

  Since T_{-b} and T are isometries, then T_A is an isometry. But isometry T_A fixes the origen and so for any $v, w \in \mathbb{R}^n$ we have

  $$\langle v, w\rangle = \langle T_A v, T_A w\rangle = \langle Av, Aw\rangle = \langle v, A^T A w\rangle.$$

  Then for all $v, w \in \mathbb{R}^n$ we have

  $$\langle v, A^T A w - w\rangle = 0,$$

  but this implies that for all $w \in \mathbb{R}^n$,

  $$A^T A w - w = 0 \Longrightarrow A^T A = I_n.$$

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Question 2– [3 point] Show that 2×2 matrix A is orthogonal if and only if

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \text{or} \qquad A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

for some θ.

Solution: First we show that Rot_θ and Ref_θ are orthogonal matrices

a) Set $A = \mathrm{Rot}_\theta$. Then

$$\begin{aligned} A^TA &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{pmatrix} \\ &= I_2. \end{aligned} \tag{2}$$

b) Set $A = \mathrm{Ref}_\theta$. Then

$$\begin{aligned} A^TA &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{pmatrix} \\ &= I_2. \end{aligned} \tag{3}$$

Hence Rot_θ and Ref_θ are orthogonal matrices.

Now we show that if A is orthogonal then $A = \mathrm{Rot}_\theta$ or $A = \mathrm{Ref}_\theta$. We present two proofs.

First proof: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an orthogonal matrix. Then $A^TA = I_2$. Therefore

$$A^TA = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So we obtain the following equations

$$\begin{aligned} a^2 + c^2 &= 1 \\ ab + cd &= 0 \\ b^2 + d^2 &= 1 \end{aligned} \tag{4}$$

Therefore there exists θ and φ such that

$$\begin{aligned} a = \cos(\theta), &\quad c = \sin(\theta) \\ b = \cos(\varphi), &\quad d = \sin(\varphi). \end{aligned} \tag{5}$$

But from $ab + cd = 0$ we obtain

$$\cos(\varphi - \theta) = \cos(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta) = 0.$$

Hence $\varphi - \theta = \pm\pi/2$ and so $\varphi = \theta + \pi/2$ or $\varphi = \theta - \pi/2$.

a) For $\varphi = \theta + \pi/2$ we have

$$b = \cos(\varphi) = \cos(\theta + \pi/2) = -\sin(\theta), \qquad d = \sin(\varphi) = \sin(\theta + \pi/2) = \cos(\theta),$$

so in this case $A = \mathrm{Rot}_\theta$.

b) For $\varphi = \theta - \pi/2$ we have

$$b = \cos(\varphi) = \cos(\theta - \pi/2) = \sin(\theta), \qquad d = \sin(\varphi) = \sin(\theta - \pi/2) = -\cos(\theta),$$

so in this case $A = \mathrm{Rof}_\theta$.

Second proof: If A is an orthogonal matrix then $A^TA = I_2$. Hence $A^{-1} = A^T$ and

$$\det(A^TA) = \det(A^T)\det(A) = \det(A)^2 = 1.$$

Therefore $\det(A) = \pm 1$. We use this information to classify orthogonal matrices. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an orthogonal matrix. Then we consider two cases:

a) $\det(A) = 1$. In this case we have

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Therefore $a = d$ and $b = -c$. But $1 = \det(A) = ad - bc = a^2 + c^2$, so $a = d = \cos(\theta)$, $c = \sin(\theta)$ and $b = -c = -\sin(\theta)$. Therefore $A = \mathrm{Rot}_\theta$.

b) $\det(A) = -1$. In this case we have

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = A^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.$$

Therefore $a = -d$ and $b = c$. But $-1 = \det(A) = ad - bc = -a^2 - c^2$, so $a = \cos(\theta)$, $d = -\cos(\theta)$ and $b = c = \sin(\theta)$. Therefore $A = \mathrm{Ref}_\theta$.

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Question 3– Prove the following equalities.

a) (1 point) $\langle u \times v, w \times z\rangle = \langle u, w\rangle\langle v, z\rangle - \langle v, w\rangle\langle u, z\rangle$.

b) (1 point) $\|u \times v\|^2 = \|u\|^2\|v\|^2 - \langle u, v\rangle^2$.

c) (2 points) $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$. This identity is called the Jacobi identity

Solution: We first recall these two identities

I $\langle v \times w, u\rangle = \langle v, w \times u\rangle$.

II (Triple product formula) $(u \times v) \times w = \langle u, w\rangle v - \langle v, w\rangle u$.

Now using these identities we proved the given identities.

a) From Identity (I), we have

$$\langle u \times v, w \times z\rangle = \langle u, v \times (w \times z)\rangle.$$

But from Identity (II) we have

$$v \times (w \times z) = -(w \times z) \times v = -\langle w, v\rangle z + \langle z, v\rangle w.$$

Therefore

$$\begin{aligned} \langle u \times v, w \times z\rangle &= \langle u, v \times (w \times z)\rangle \\ &= \langle u, -\langle w, v\rangle z + \langle z, v\rangle w\rangle \\ &= \langle u, w\rangle\langle v, z\rangle - \langle v, w\rangle\langle u, z\rangle. \end{aligned} \tag{6}$$

b) We have shown that

$$\langle u \times v, w \times z\rangle = \langle u, w\rangle\langle v, z\rangle - \langle v, w\rangle\langle u, z\rangle.$$

Therefore

$$||u \times v||^2 = \langle u \times v, u \times v\rangle = \langle u, u\rangle\langle v, v\rangle - \langle v, u\rangle\langle u, v\rangle = ||u||^2||v||^2 - \langle u, v\rangle^2.$$

c) We recall again the triple product formula

$$(u \times v) \times w = \langle u, w\rangle v - \langle v, w\rangle u.$$

Therefore

$$\begin{aligned} (u \times v) \times w &= \langle u, w\rangle v - \langle v, w\rangle u \\ (v \times w) \times u &= \langle v, u\rangle w - \langle w, u\rangle v \\ (w \times u) \times v &= \langle w, v\rangle u - \langle u, v\rangle w. \end{aligned} \tag{7}$$

Therefore

$$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0.$$

Notice that

$$u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = -\left((u \times v) \times w + (v \times w) \times u + (w \times u) \times v\right) = 0.$$