MAT 2355, Fall 2014
Assignment 3-Solution

(10 points)

Instructor: Mohammad Bardestani

Question 1– [3 points] Prove that Rot_θ has a *non-zero real* eigenvalue if and only if $\mathrm{Rot}_\theta = \pm I$.

Solution: First we calculate the characteristic polynomial of Rot_θ. We have

$$\det(\lambda I - \mathrm{Rot}_\theta) = \det\begin{pmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{pmatrix} = \lambda^2 - (2\cos\theta)\lambda + \cos^2\theta + \sin^\theta = \lambda^2 - (2\cos\theta)\lambda + 1.$$

Therefore Rot_θ has a real eigenvalue if and only if $4\cos^2\theta - 4 \geq 1$. This shows that Rot_θ has a real eigenvalue if and only if $\cos^2\theta = 1$. Therefore $\theta = k\pi$, for some $k \in \mathbb{Z}$, which shows that Rot_θ has a *non-zero real* eigenvalue if and only if $\mathrm{Rot}_\theta = \pm I$.

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Question 2– [3 point] Let v be a unit vector with a unit normal vector N. Let $\ell = P + [v]$ and $\mathfrak{m} = Q + [v]$ be two parallel lines. Show that

$$\Omega_\ell \circ \Omega_\mathfrak{m} = T_w, \qquad \text{where} \qquad w = 2\langle P - Q, N\rangle N.$$

Solution: Let Q' be the foot of P on $\mathfrak{m}$. Then we have

$$\Omega_\ell \circ \Omega_\mathfrak{m} = T_{2(P-Q')}.$$

But (look at the formula for finding the foot)

$$Q' = P - \langle P - Q, N\rangle N.$$

Hence

$$2(P - Q') = 2(P - (P - \langle P - Q, N\rangle N)) = 2\langle P - Q, N\rangle N.$$

Therefore

$$\Omega_\ell \circ \Omega_\mathfrak{m} = T_{2(P-Q')} = T_{2\langle P-Q,N\rangle N} = T_w.$$

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Question 3– [4 points] Show that two distinct reflections Ω_ℓ and $\Omega_\mathfrak{m}$ commute (i.e., $\Omega_\ell \circ \Omega_\mathfrak{m} = \Omega_\mathfrak{m} \circ \Omega_\ell$), if and only if $\ell \perp \mathfrak{m}$.

Solution: Let $\ell \perp \mathfrak{m}$, and assume that $P \in \ell \cap \mathfrak{m}$. Then

$$\begin{aligned} \Omega_\ell &= T_P \circ \mathrm{Ref}_\theta \circ T_{-P} \\ \Omega_\mathfrak{m} &= T_P \circ \mathrm{Ref}_{\pi/2+\theta} \circ T_{-P}. \end{aligned} \tag{1}$$

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that sends X to $-X$. This map is a linear transformation and its matrix with respect to the standard basis is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that

$$\begin{aligned} \mathrm{Ref}_\theta \circ \mathrm{Ref}_{\pi/2+\theta} &= \mathrm{Rot}_{2(\theta-(\pi/2+\theta))} = \mathrm{Rot}_{-\pi} = S \\ \mathrm{Ref}_{\pi/2+\theta} \circ \mathrm{Ref}_\theta &= \mathrm{Rot}_{2((\pi/2+\theta)-\theta)} = \mathrm{Rot}_\pi = S. \end{aligned} \tag{2}$$

Therefore

$$\begin{aligned} \Omega_\ell \circ \Omega_\mathfrak{m} &= T_P \circ \mathrm{Ref}_\theta \circ \mathrm{Ref}_{\pi/2+\theta} \circ T_{-P} = T_P \circ S \circ T_{-P} \\ \Omega_\mathfrak{m} \circ \Omega_\ell &= T_P \circ \mathrm{Ref}_{\pi/2+\theta} \circ \mathrm{Ref}_\theta \circ T_{-P} = T_P \circ S \circ T_{-P}. \end{aligned} \tag{3}$$

So

$$\Omega_\ell \circ \Omega_\mathfrak{m} = \Omega_\mathfrak{m} \circ \Omega_\ell.$$

Conversely assume that $\Omega_\ell \circ \Omega_\mathfrak{m} = \Omega_\mathfrak{m} \circ \Omega_\ell$. first we show that $\ell \nparallel \mathfrak{m}$. Suppose $\ell \parallel \mathfrak{m}$. Let Q be a point on ℓ and assume that P is the foot of Q on $\mathfrak{m}$. Then

$$\begin{aligned} \Omega_\ell \circ \Omega_\mathfrak{m} &= T_{2(Q-P)} \\ \Omega_\mathfrak{m} \circ \Omega_\ell &= T_{2(P-Q)}. \end{aligned} \tag{4}$$

Since $P \neq Q$, then $T_{2(Q-P)} \neq T_{2(P-Q)}$, which is a contradiction, since we assumed that $\Omega_\ell \circ \Omega_\mathfrak{m} = \Omega_\mathfrak{m} \circ \Omega_\ell$. Hence $\ell \nparallel \mathfrak{m}$. Assume $P \in \ell \cap \mathfrak{m}$. Then

$$\begin{aligned} \Omega_\ell &= T_P \circ \mathrm{Ref}_\theta \circ T_{-P} \\ \Omega_\mathfrak{m} &= T_P \circ \mathrm{Ref}_\varphi \circ T_{-P}. \end{aligned} \tag{5}$$

Therefore

$$\begin{aligned} \Omega_\ell \circ \Omega_\mathfrak{m} &= T_P \circ \mathrm{Ref}_\theta \circ \mathrm{Ref}_\varphi \circ T_{-P} = T_P \circ \mathrm{Rot}_{2(\theta-\varphi)} \circ T_{-P} \\ \Omega_\mathfrak{m} \circ \Omega_\ell &= T_P \circ \mathrm{Ref}_\varphi \circ \mathrm{Ref}_\theta \circ T_{-P} = T_P \circ \mathrm{Rot}_{2(\varphi-\theta)} \circ T_{-P}. \end{aligned} \tag{6}$$

But $\Omega_\ell \circ \Omega_\mathfrak{m} = \Omega_\mathfrak{m} \circ \Omega_\ell$, so $\mathrm{Rot}_{2(\theta-\varphi)} = \mathrm{Rot}_{2(\varphi-\theta)}$. This implies that for some $\kappa \in \mathbb{Z}$

$$-\sin(2(\varphi-\theta)) = \sin(2(\theta-\varphi)) = \sin(2(\varphi-\theta)) \Rightarrow \sin(2(\varphi-\theta)) = 0 \Rightarrow 2(\varphi-\theta) = \kappa\pi \Rightarrow \varphi = (\kappa\pi)/2 + \theta.$$

Therefore $\ell \perp \mathfrak{m}$.