MAT 2355, Fall 2014
Assignment 3-Solution

(10 points)

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Question 1– [3 points] Let $C \subseteq \mathbb{R}^2$ be the circle with center P and radius r. Prove that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry then $T(C)$ is the circle with center $T(P)$ and radius r.

Solution: Let us denote $C_r(P)$, the circle with center P and radius r. By definition, $C_r(P)$ is the set of all points with the distance r from P, i.e,

$$C_r(P) := \{X \in \mathbb{R}^2 : d(X, P) = r\}.$$

We want to show that

$$T(C_r(P)) = C_r(T(P)). \tag{1}$$

Let $X \in C_r(P)$, then

$$d(T(X), T(P)) = d(X, P) = r \Longrightarrow T(X) \in C_r(T(P)).$$

Hence

$$T(C_r(P)) \subseteq C_r(T(P)).$$

With the same method we have

$$T^{-1}(C_r(T(P))) \subseteq C_r(T^{-1} \circ T(P)) = C_r(P) \Longrightarrow C_r(T(P) \subseteq T(C_r(P)).$$

Therefore we obtain Equation (1).

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Question 2– [1 point] Show that reflection in the x axis in $\mathbb{R}^2$ is given by the formula

$$T(x, y) = (x, -y).$$

Solution: Let $e_1 = (1, 0)$. Then $\ell = [e_1]$ is the x axis. So $e_2 = (0, 1)$ is a unit normal vector, and then for any $X = (x, y) \in \mathbb{R}^2$ we have

$$\Omega_\ell(X) = X - 2\langle X, N\rangle N = (x, y) - 2ye_2 = (x, -y).$$

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Question 3– [1 point] Show that the following map is a translation.

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x + 2, y - \pi),$$

Solution: Let $v = (2, -\pi)$, then $T = T_v$, and so T is a translation map.

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Question 4– [5 points] Show

1. Show that ISO($\mathbb{R}^2$) is a group, and find
   (a) The identity element.
   (b) Inverse of Ω_ℓ.
   (c) Inverse of translation T_v.
2. The set of all translations of $\mathbb{R}^2$ is denoted by TRANS($\mathbb{R}^2$). Show that TRANS($\mathbb{R}^2$) is an abelian group.

Solution: For two isometries $T_1, T_2 \in$ ISO($\mathbb{R}^2$), we proved that $T_1 \circ T_2 \in$ ISO($\mathbb{R}^2$), which shows the closure axiom. Also it is evident that for $T_i \in$ ISO($\mathbb{R}^2$), $1 \leq i \leq 3$,

$$(T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3),$$

which proves the associativity axiom. Notice that the identity map, denoted by I, is an isometry. Moreover for all isometry $T \in$ ISO($\mathbb{R}^2$),

$$T \circ I = I \circ T = T,$$

Hence we have the identity axiom. Also we have proved that for any $T \in$ ISO($\mathbb{R}^2$), the inverse of T, denoted by T^{-1}, is an isometry and

$$T \circ T^{-1} = T^{-1} \circ T = I,$$

which verifies the inverse axiom. Therefore ISO($\mathbb{R}^2$) is a group. We showed that $\Omega_\ell \circ \Omega_\ell = I$, and hence the inverse of Ω_ℓ is Ω_ℓ. Also we have seen that $T_v \circ T_{-v} = I$, and so the inverse of T_v is T_{-v}.

With the same method, used in the first part, we can show that TRANS($\mathbb{R}^2$) is a group. We now prove that TRANS($\mathbb{R}^2$) is an abelian group. For two translation $T_v, T_w \in$ TRANS($\mathbb{R}^2$) we have

$$T_v \circ T_w = T_{v+w} = T_w \circ T_v,$$

and so TRANS($\mathbb{R}^2$) is an abelian group.