MAT 2355, Fall 2014
Assignment 2-Solution

(10 points)

Instructor: Mohammad Bardestani

Question 1– [2 points] Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that preserves length; that is, $||T(v)|| = ||v||$ for all $v \in \mathbb{R}^n$. Prove that for any $v, w \in \mathbb{R}^n$ we have

$$\langle T(v), T(w) \rangle = \langle v, w \rangle.$$

Solution: First let us recall the following identity:

$$\langle v, w \rangle = \frac{1}{2}\left(||v||^2 + ||w||^2 - ||v - w||^2\right). \tag{1}$$

Hence

$$\begin{aligned} \langle T(v), T(w) \rangle &= \frac{1}{2}\left(||T(v)||^2 + ||T(w)||^2 - ||T(v) - T(w)||^2\right) \\ &= \frac{1}{2}\left(||T(v)||^2 + ||T(w)||^2 - ||T(v - w)||^2\right) \qquad \text{since } T \text{ is linear} \\ &= \frac{1}{2}\left(||v||^2 + ||w||^2 - ||v - w||^2\right) \qquad \text{since } T \text{ preserves length} \\ &= \langle v, w \rangle. \end{aligned} \tag{2}$$

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Question 2– [2 points] Let $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$ be an orthonormal basis. Prove that for any $v \in \mathbb{R}^n$ we have

$$||v||^2 = \langle v, v_1 \rangle^2 + \cdots + \langle v, v_n \rangle^2.$$

Solution: Since $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$ is an orthonormal basis, then for any vector $v \in \mathbb{R}^n$ we have

$$v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n = \sum_{i=1}^{n} \langle v, v_i \rangle v_i.$$

Therefore

$$||v||^2 = \langle v, v \rangle = \langle v, \sum_{i=1}^{n} \langle v, v_i \rangle v_i \rangle = \sum_{i=1}^{n} \langle v, \langle v, v_i \rangle v_i \rangle = \sum_{i=1}^{n} \langle v, v_i \rangle^2.$$

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Question 3– [1 point] Let $\ell = (0,1) + [(2,1)]$ be a line in $\mathbb{R}^2$. Find the intersection point of the line through $X = (3,4)$ perpendicular to ℓ.

Solution: From the formula of the foot of X on ℓ, denoted by F (which is the same as the intersection point of the line through $X = (3,4)$ perpendicular to ℓ), is equal to

$$F = X - \langle X - P, N\rangle N,$$

where N is a unit normal vector and P is any point in ℓ. Pick $P = (0,1)$, and $N = 1/\sqrt{5}(-1,2)$. Then we have $X - P = (3,3)$, $\langle X - P, N\rangle = 3/\sqrt{5}$

$$F = (3,4) - 3/5(-1,2) = (18/5, 14/5).$$

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Question 4– [2 points] Let X be a point in $\mathbb{R}^2$ and ℓ a line. Let F be the foot of X on ℓ, then show that F is the point of ℓ nearest to X. In other words if Q is any point of ℓ then

$$||X - Q|| \geq ||X - F||.$$

Solution: Let $Q \in \ell$. Notice that $F - Q \perp X - F$. Therefore by Pythagoras theorem (see Theorem 2 of Week 2) we have

$$||X - Q||^2 = ||(X - F) + (F - Q)||^2 = ||X - F||^2 + ||F - Q||^2 \geq ||X - F||^2.$$

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Question 5– [2 points] Let ℓ_1 and ℓ_2 be parallel lines. Let

$$\ell_3 := \{1/2(X_1 + X_2) : X_1 \in \ell_1, \quad X_2 \in \ell_2\}.$$

Prove that ℓ_3 is a line parallel to ℓ_1, ℓ_2.

Solution: Notice that ℓ_1 and ℓ_2 are parallel. Then we can write $\ell_1 = P_1 + [v]$ and $\ell_2 = P_2 + [v]$. Therefore for $X_i = P_i + t_i v \in \ell_i$, $i = 1,2$ we have

$$1/2(X_1 + X_2) = 1/2(P_1 + P_2) + 1/2(t_1 + t_2)v.$$

This implies that

$$\ell_3 = 1/2(P_1 + P_2) + [v],$$

and hence ℓ_3 is a line parallel to ℓ_1, ℓ_2.

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Question 6– [1 points] For given line ℓ, let $\Omega_\ell(X)$ be the reflection of point X with respect to line ℓ. Show that $\Omega_\ell(X) = X$ if and only if $X \in \ell$.

Solution: Let $\ell = P + [v]$. Let $X = P + tv \in \ell$. From this we have

$$tv = X - P \perp N \Longrightarrow \langle X - P, N\rangle = 0,$$

and so

$$\Omega_\ell(X) = X - 2\langle X - P, N\rangle N = X.$$

Conversely assume $\Omega_\ell(X) = X$, then

$$X = X - 2\langle X - P, N\rangle N \Longrightarrow \langle X - P, N\rangle N = 0 \Longrightarrow \langle X - P, N\rangle = 0.$$

Then from Theorem 3 of the Week 2, we can conclude that $X \in \ell$.