MAT 2355, Fall 2014
Assignment 1-Solution

(10 points)

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Question 1– [2 points] Let P, Q be two points in $\mathbb{R}^2$. If $0 < t < 1$ and $X = (1-t)P + tQ$, show that

$$\frac{d(P,X)}{d(X,Q)} = \frac{t}{1-t}.$$

Use this to find the point X that divides the segment PQ in the ratio $r : s$.

Solution: By definition we have

$$\begin{aligned} \frac{d(P,X)}{d(X,Q)} &= \frac{||X-P||}{||X-Q||} = \frac{||(1-t)P+tQ-P||}{||(1-t)P+tQ-Q||} \\ &= \frac{||-tP+tQ||}{||(1-t)P+(t-1)Q||} = \frac{t||Q-P||}{(1-t)||(Q-P||} \\ &= \frac{t}{1-t}. \end{aligned} \tag{1}$$

We use this to find the point X that divides the segment PQ in the ratio $r : s$. Hence we need to find t such that

$$\frac{t}{1-t} = \frac{r}{s} \Longrightarrow st = (1-t)r = r - tr \Longrightarrow t(s+t) = r \Longrightarrow t = \frac{r}{r+s}.$$

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Question 2– [2 points] For $v, w \in \mathbb{R}^n$ show that

$$\langle v, w\rangle = \frac{1}{2}\left(||v||^2 + ||w||^2 - ||v-w||^2\right)$$

Solution: For any vector $v \in \mathbb{R}^n$ we have $\langle v, v\rangle = ||v||^2$. Moreover notice that

$$\langle v-w, v-w\rangle = \langle v,v\rangle + \langle v,-w\rangle + \langle -w,v\rangle + \langle -w,-w\rangle = ||v||^2 + ||w||^2 - 2\langle v,w\rangle. \tag{2}$$

Hence by Equation (2) we have

$$\begin{aligned} \frac{1}{2}\left(||v||^2 + ||w||^2 - ||v-w||^2\right) &= \frac{1}{2}\left(||v||^2 + ||w||^2 - \langle v-w, v-w\rangle\right) \\ &= \frac{1}{2}\left(||v||^2 + ||w||^2 - \left(||v||^2 + ||w||^2 - 2\langle v,w\rangle\right)\right) \\ &= \langle v, w\rangle. \end{aligned} \tag{3}$$

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Question 3– [2 points] Let $X \in \mathbb{R}^2$ be a point and let ℓ be a line in plane. Prove that there exists a unique line $\mathfrak{m}$ through X perpendicular to ℓ.

Solution: Let $\ell = P + [v]$. Then by definition $\mathfrak{m} = X + [N]$ is perpendicular to ℓ when $N \perp v$. So to show that this perpendicular line is unique, we need to show that any two vectors in $v^\perp$ are linearly dependent. This in particular implies that for $N_1, N_2 \in v^\perp$, the lines $X + [N_1]$ and $X + [N_2]$ are the same lines.

Let us assume that $N_1, N_2 \in v^\perp$ are linearly independent. Then $\{N_1, N_2\}$ is a basis for $\mathbb{R}^2$. Since $\mathrm{Span}\{N_1, N_2\} = \mathbb{R}^2$ and by our assumption $v \perp N_i$, for $i = 1, 2$, we can conclude that $v = 0$ which is a contradiction since $v \neq 0$.

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Question 4– [1 point] Let $v_1, \ldots, v_k \in \mathbb{R}^n$. Show that if $v \perp v_i$, for $1 \leq i \leq k$, the $v \in W^\perp$, where W is the subspace of $\mathbb{R}^n$ generated by $v_1, \ldots, v_k$.

Solution: By the definition of subspace generated by $v_1, \ldots, v_k$, denoted by $W = \mathrm{Span}\{v_1, \ldots, v_k\}$, we have

$$W = \{a_1 v_1 + \cdots + a_k v_k : a_i \in \mathbb{R}\}.$$

For any $w = a_1 v_1 + \cdots + a_k v_k \in W$, since $v \perp v_i$ for all $1 \leq i \leq k$, we have

$$\begin{aligned} \langle v, w \rangle &= \langle v, a_1 v_1 + \cdots + a_k v_k \rangle \\ &= a_1 \langle v, v_1 \rangle + \cdots + a_k \langle v, v_k \rangle \\ &= 0. \end{aligned} \tag{4}$$

Therefore we conclude that $v \in W^\perp$.

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Question 5– [2 points] Let $v_1 = (3, 1, 1)$, $v_2 = (-1, 2, 1)$ and $v_3 = (-1/2, -2, 7/2)$ be three vectors in $\mathbb{R}^3$. Show that $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Also express $y = (6, 1, -8)$ as a linear combination of v_1, v_2, v_3.

Solution: To show that $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$, we need to show that $v_i \perp v_j$ for all $1 \le i \ne j \le 3$. We have

$$\begin{aligned} \langle v_1, v_2 \rangle &= -3 + 2 + 1 = 0. \\ \langle v_1, v_3 \rangle &= -3/2 - 2 + 7/2 = 0. \\ \langle v_2, v_3 \rangle &= 1/2 - 4 + 7/2 = 0. \end{aligned} \tag{5}$$

Hence $\{v_1, v_2, v_3\}$ is an orthogonal basis and then $\{w_1 = v_1/\|v_1\|, w_2 = v_2/\|v_2\|, w_3 = v_3/\|v_3\|\}$ is an orthonormal basis. Therefore we have

$$\begin{aligned} y &= \langle y, w_1 \rangle w_1 + \langle y, w_2 \rangle w_2 + \langle y, w_3 \rangle w_3 \\ &= \frac{\langle y, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle y, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \frac{\langle y, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3. \end{aligned} \tag{6}$$

We have

$$\begin{array}{ccc} \langle y, v_1 \rangle = 11, & \langle y, v_2 \rangle = -12, & \langle y, v_3 \rangle = -33 \\ \langle v_1, v_1 \rangle = 11, & \langle v_2, v_2 \rangle = 6, & \langle v_3, v_3 \rangle = 33/2 \end{array}$$

Hence

$$y = v_1 - 2v_2 - 2v_3.$$

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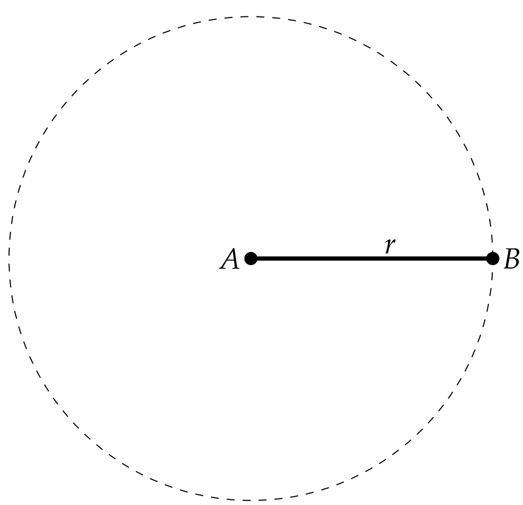
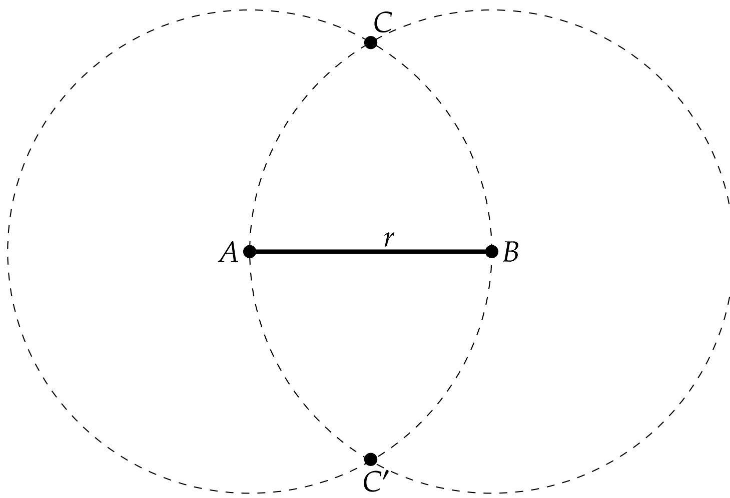
Question 6– [1 point] Given points A and B, find, with a Euclidean compass alone, the midpoint M of segment AB. Prove your method.

Solution:
Step I: Draw a circle with the center A and the radios $r = d(A, B)$.

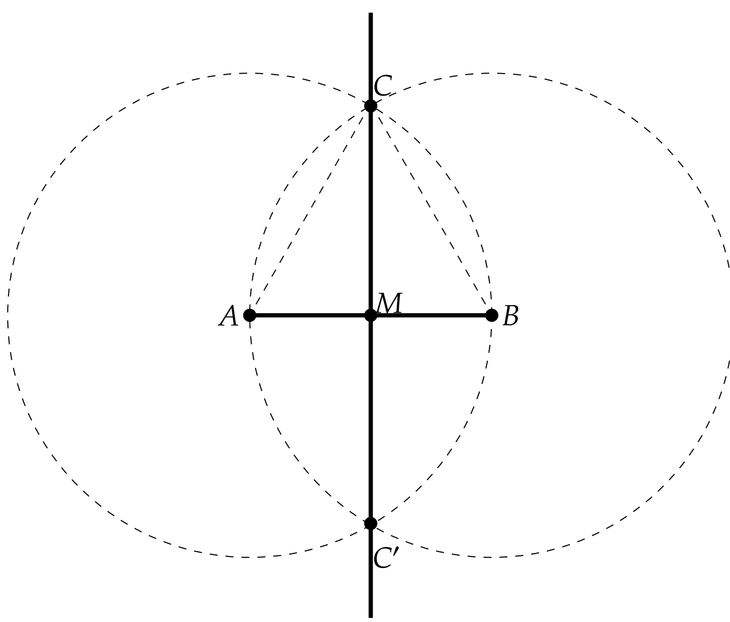


Step II: Draw a circle with the center B and the radios $r = d(A, B)$.



These two circles intersect at two points C and C'.

Step III: Connect two points C and C' which this line intersects the segment AB at a point M. We claim that this point is the midpoint of the segment AB.



Notice that $d(C, A) = d(C, B) = d(A, B) = r$, hence $\angle CAB = \angle CBA = 60°$. Hence the triangle $\triangle CAM$ is congruent to the triangle $\triangle CBM$, and hence $d(A, M) = d(B, M)$.