This test paper has 4 questions and total of 20 marks. It can not be taken from the examination room. Calculators are allowed. SHOW ALL YOUR WORK. Duration: 50 minutes.

NAME: SOLUTIONS

STUDENT NUMBER:

1. [**5 Marks**]Let $w = 1 + 4i$ and let $z = -3 + 2i$. Compute the following:

- $w + z$ $\quad (1+4i) + (-3+2i) = (1-3) + (4+2)i = -2+6i$
- $z - w$ $\quad (-3+2i) - (1+4i) = (-3-1) + (2-4)i = -4 + -2i$
- wz $\quad (1+4i)(-3+2i) = -3+2i-12i+8i^2 = -11-10i$
- z/w

$$\frac{-3+2i}{1+4i} = \frac{-3+2i}{1+4i}\,\frac{1-4i}{1-4i}$$

$$= \frac{(-3+2i)(1-4i)}{1+16} = \frac{-3+12i+2i-8i^2}{17}$$

$$= \frac{5}{17} + \frac{14}{17}i$$

2. [**5 Marks**]Let $p(x)$ be the polynomial $p(x) = x^2 + x + 1$.

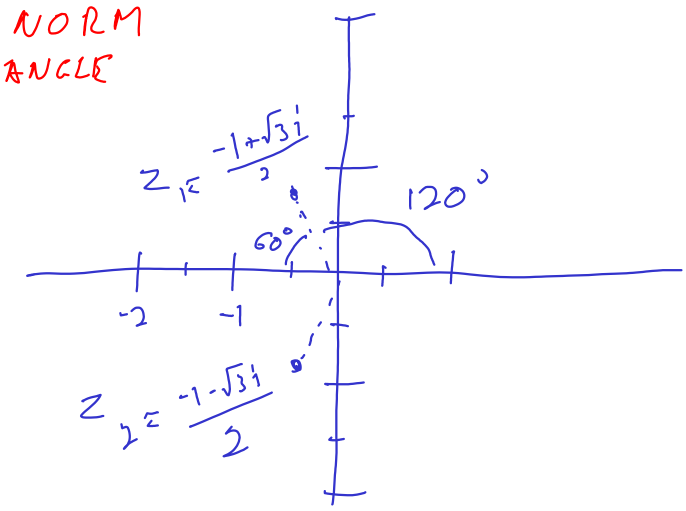
- Use the quadratic formula to find the two complex roots of $p(x)$.
- Roughly plot each root of $p(x)$ in the complex plane and express it in polar form (you may use degrees or radians). (Useful fact: $\cos(2\pi/3) = \cos(120°) = -\frac{\sqrt{3}}{2}$.)
- Let z_1 be the root of $p(x)$ with a **positive** imaginary component (i.e. $z_1 = a + bi$, with $b \in \mathbb{R}$ and $b > 0$.) Using de Moivre's theorem find which of the below (there may be more than one) are 6^{th} roots of z_1.

(a) $\text{cis}(\frac{\pi}{9}) = \text{cis}(20°)$.

~~(b) $\sqrt{2}\,\text{cis}(\frac{4\pi}{9}) = \sqrt{2}\,\text{cis}(80°)$~~ WRONG NORM

~~(c) $\text{cis}(\frac{8\pi}{9}) = \text{cis}(160°)$~~ WRONG ANGLE

(d) $\text{cis}(\frac{10\pi}{9}) = \text{cis}(200°)$



$$z_1 = \frac{-1 + \sqrt{1-4}}{2} = \frac{-1 + \sqrt{3}\,i}{2}$$

$$z_2 = \frac{-1 - \sqrt{3}\,i}{2}$$

$$z_1 = 1 \text{ cis } 120° = 1 \text{ cis } (2\pi/3)$$

$$z_2 = 1 \text{ cis } 240° = 1 \text{ cis } (4\pi/3)$$

$$z_1^{1/6} = 1^{1/6} \text{ cis } \left(\frac{1}{6}(120° + j\,360°)\right) \qquad j = 0, 1, 2, 3, 4, 5$$

$$= 1 \text{ cis } (20° + j\,60°)$$

THEY ARE

$1 \text{ cis } (20°) = 1 \text{ cis } \pi/9$

$1 \text{ cis } (80°) = 1 \text{ cis } 4\pi/9$

$1 \text{ cis } (140°) = 1 \text{ cis } 7\pi/9$

$1 \text{ cis } (200°) = 1 \text{ cis } 10\pi/9$

$1 \text{ cis } (260°) = 1 \text{ cis } 13\pi/9$

$1 \text{ cis } (320°) = 1 \text{ cis } 16\pi/9$

3. [**5 Marks**]Let $\vec{w} = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ be vectors in $\mathbb{R}^3$. Compute the following:

- $\vec{w} + \vec{v}$
- $|\vec{w}|$, $|\vec{v}|$
- $\vec{w} \cdot \vec{v}$, $\vec{v} \cdot \vec{e}_3$
- The angle (in degrees or radians) between $\vec{v}$ and $\vec{e}_3$. (Useful fact: $\sqrt{50} = \sqrt{2}\sqrt{25}$.)
- $\text{proj}_{\vec{v}}\vec{w}$, $\text{orth}_{\vec{v}}\vec{w}$

$$\vec{w} + \vec{v} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}4\\4\\6\end{pmatrix}$$

$$|\vec{w}| = \sqrt{1^2+0^2+1^2} = \sqrt{2} \qquad |\vec{v}| = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$$

$$\vec{w}\cdot\vec{v} = 3+0+5 = 8 \qquad \vec{v}\cdot\vec{e}_3 = 0+0+5 = 5$$

LET θ BE THE ANGLE BETWEEN $\vec{v}$ AND $\vec{e}_3$

$$\cos\theta = \frac{\vec{v}\cdot\vec{e}}{|\vec{v}||\vec{e}|} = \frac{5}{5\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}} \quad \text{SO} \quad \theta = 45^\circ = \pi/4$$

$$\text{proj}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|^2}\vec{v} = \frac{8}{50}\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}12/25\\16/25\\4/5\end{pmatrix}$$

$$\text{ORTH}_{\vec{v}}\vec{w} = \vec{w} - \text{proj}_{\vec{v}}w = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}12/25\\16/25\\4/5\end{pmatrix} = \begin{pmatrix}13/25\\-16/25\\1/5\end{pmatrix}$$

4. [**5 Marks**]Let $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ be vectors in $\mathbb{R}^3$.

$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Expand the expression

$$\vec{u} \cdot (\vec{v} \times \vec{u})$$

  and simplify it as much as possible (it should simplify a lot.)

- What did you just prove? (Multiple choice, one answer.)
  - (a) $\vec{u}$ is orthogonal to $\vec{u} \times \vec{v}$.
  - (b) $\vec{u}$ is parallel to $\vec{u} \times \vec{v}$.
  - (c) $\vec{u} \times \vec{v}$ always equals $\vec{0}$.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} ec - bf \\ fa - dc \\ db - ea \end{pmatrix} = aec - abf + bfa - bdc + cdb - cea$$

$$= ace - abf + abf - bcd + bcd - ace$$

$$= ace - ace + abf - abf + bcd - bcd$$

$$= 0$$

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