LECTURE 3: GRAPHS AND LEVEL CURVES

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Theorem 1. *The plane passing through the point $P_0 = (x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation:*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \quad \text{or} \quad ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

Remark 1. The general equation for a plane is given by $ax + by + cz = d$, then a normal vector can be $\mathbf{n} = \langle a, b, c\rangle$.

Traces

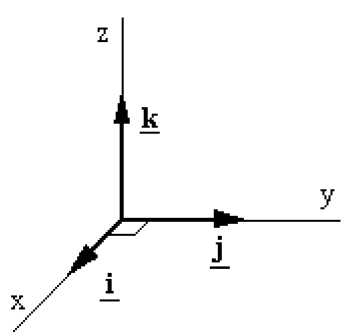
Definition 1. A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy-trace, the yz-trace, and the xz-trace.

Remark 2. .

I. The normal direction of yz-plane (that is, the direction of x-axis) is the vector $\mathbf{i} = \langle 1, 0, 0\rangle$.

II. The normal direction of xz-plane (that is, the direction of y-axis) is the vector $\mathbf{j} = \langle 0, 1, 0\rangle$.

III. The normal direction of xy-plane (that is, the direction of z-axis) is the vector $\mathbf{k} = \langle 0, 0, 1\rangle$.



Theorem 2. *The general equation of a surface can be implicitly given by $F(x, y, z) = 0$ be the equation of a surface, then*

a. *The trace in the plane $x = a$ is given by the equation $F(a, y, z) = 0$.*

b. *The trace in the plane $y = b$ is given by the equation $F(x, b, z) = 0$.*

c. *The trace in the plane $z = c$ is given by the equation $F(x, y, c) = 0$.*

Example 1. The equation of the sphere with center at 0 and radius 4 is given by: $x^2 + y^2 + z^2 = 4^2$. Then

a. The trace in the plane $x = 1$ is given by the equation $1^2 + y^2 + z^2 = 4^2$, that is, $y^2 + z^2 = 15$, which is a circle.

b. The trace in the plane $z = 2$ is given by the equation $x^2 + y^2 + 2^2 = 4^2$, that is, $x^2 + y^2 = 12$, which is also a circle.

Example 2. The surface defined by the equation $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is called an elliptic paraboloid. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

The equation for the surface is $\frac{x^2}{16} + \frac{y^2}{4} = z$. Then

- It's clear that $z \geq 0$
  - when $z = 0$, we must have $x = y = 0$.
  - when $z_0 > 0$, then the trace in the horizontal plane $z = z_0$ satisfies the equation $\frac{x^2}{16} + \frac{y^2}{4} = z_0$, which is an ellipse.

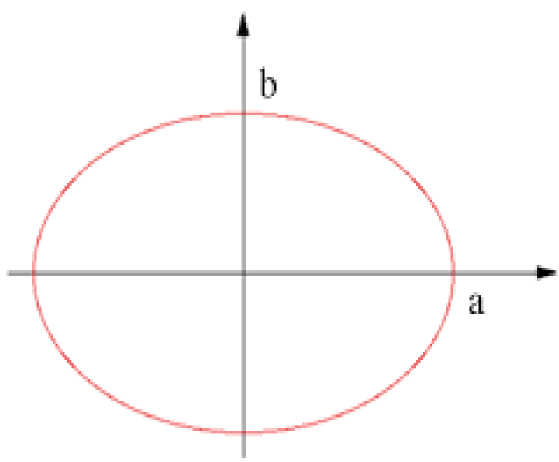


FIGURE 1. The graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- The trace in the vertical plane $x = x_0$ is the parabola $z = \frac{x_0^2}{16} + \frac{y^2}{4}$.

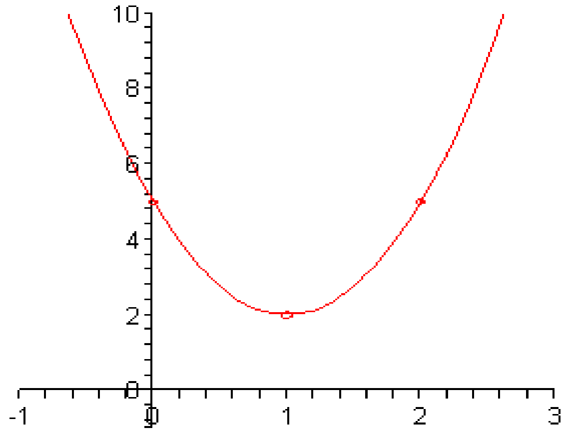


FIGURE 2. The graph of a parabola

- The trace in the vertical plane $y = y_0$ is the parabola $z = \dfrac{x^2}{16} + \dfrac{y_0^2}{4}$.

To graph the surface, we sketch the xz-trace $z = \dfrac{x^2}{16}$ (setting $y = 0$), and the yz-trace $z = \dfrac{y^2}{4}$ (setting $x = 0$). Now we graph an elliptic trace $\dfrac{x^2}{16} + \dfrac{y^2}{4} = z_0$ in the plane $z = z_0$.

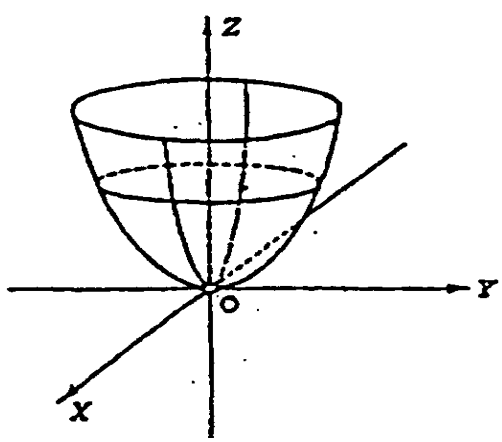


FIGURE 3. The graph of an elliptic paraboloid

Functions of two variables

In general, functions of two variables are written *explicitly* in the form

$$z = f(x, y)$$

or *implicitly* in the form

$$F(x, y, z) = 0.$$

Definition 2. A function $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in $\mathbb{R}$. The set D is called the domain of the function f. The range of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain D.

Example 3. A polynomial $f(x, y) = x^2y - 2xy - xy^2$ is a function of two variables x, y. Such polynomial is defined for all x and y, so its domain is $\mathbb{R}^2$.

Example 4. Find the domain and range of the function $g(x, y) = \sqrt{4 - x^2 - y^2}$.

Because g involves a square root, so $4 - x^2 - y^2 \geq 0$, that is, $x^2 + y^2 \leq 4$. Therefore, the domain of g is the set $\{(x, y) : \ x^2 + y^2 \leq 4\}$, which is the set of points on or within the circle of radius 2 centered at the origin in the xy-plane.

When (x, y) varies over the disk $\{(x, y) : \ x^2 + y^2 \leq 4$, then $4 - x^2 - y^2$ can vary from 0 to 4, which implies that $g(x, y) = \sqrt{4 - x^2 - y^2}$ can vary from 0 to 2. Therefore, the range of g is the closed interval $[0, 2]$.

Example 5. Find the domain of the function $g(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$.

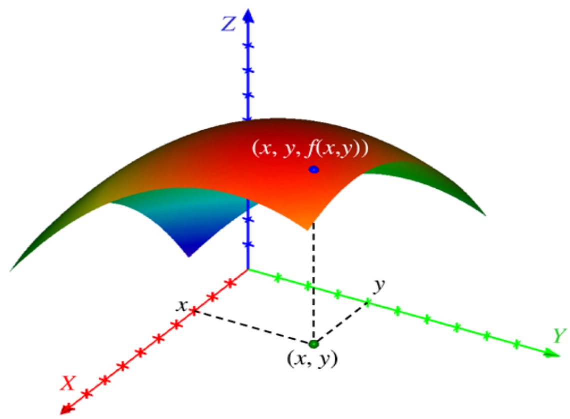
Because g involves a square root and a rational, then $x^2 + y^2 - 1 > 0$, that is, $x^2 + y^2 > 1$. Therefore, the domain of g is the set $\{(x, y) : \ x^2 + y^2 > 1\}$, which is the set of points outside the circle of radius 1 centered at the origin in the xy-plane.

Example 6. Find the domain of the function $f(x, y) = \ln(\sin(xy))$.

Because f contains ln function, then $\sin(xy) > 0$. So the domain of f is the set $\{(x, y) : \ \sin(xy) > 0\}$.

Graphs of functions of two variables

Definition 3. The graph of a function f of two variables is the set of points (x, y, z) that satisfy the equation $z = f(x, y)$, where (x, y) varies over the domain of the function.

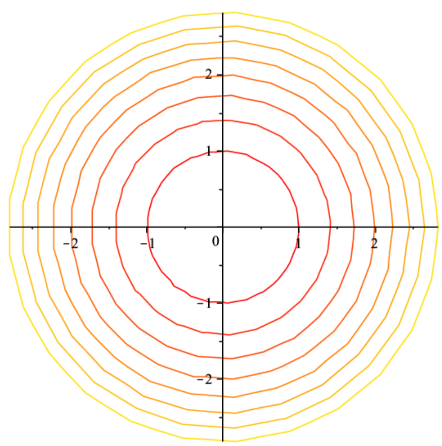


Level curves

Definition 4. Given a surface $z = f(x, y)$, for any z_0, the curve $f(x, y) = z_0$ in the xy-plane is called a level curve of the surface $z = f(x, y)$ with level z_0.

Remark 3. A level curve of a surface $z = f(x, y)$ with level z_0 is just the projection of the trace of a surface $z = f(x, y)$ in the plane $z = z_0$ onto the xy-plane.

Example 7. The level curves of a sphere $x^2 + y^2 + z^2 = 5^2$ are just circles centered at the origin.



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