CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number		Section(s)
Mathematics	208/4		All except EC
Examination	Date	Time	Pages
Final	April 2011	3 Hours	3

Instructors	Course Examiner
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FORMULAE:
$A = P(1+i)^n$, $A = Pe^{rt}$, $FV = PMT\,\dfrac{(1+i)^n - 1}{i}$, $PV = PMT\,\dfrac{1-(1+i)^{-n}}{i}$

Special Instructions:

- ▷ Answer all questions.
- ▷ **Only approved calculators are allowed.**

MARKS

[10] **1.** The marketing research department for a company that manufactures and sells notebook computers established the following price-demand, revenue functions and cost functions:

$$p(x) = 2,000 - 60x$$
$$R(x) = x(2,000 - 60x)$$
$$C(x) = 4,000 + 500x$$

where p(x) is the wholesale price in dollars at which x thousand computers can be sold, and $R(x)$ and $C(x)$ are in thousands of dollars. The functions $R(x)$ and $C(x)$ have domain $1 \leq x \leq 25$.

(A) What is the wholesale price per computer (to the nearest dollar) that produces the maximum revenue?

(B) Find the break-even points.

(C) For what values of x will a loss occur? A profit?

[10] **2.** Solve for x in the following equations:

(A) $e^{12} = (e^4)^x e^{x^2}$

(B) $(27)^{2x} = (3)^{x^2-7}$

(C) $\log_b x = \frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + 2\log_b 2$

(D) $\log_a x + \log_a(x-4) = \log_a(21)$

(E) $\log_8(x+6) = 1 - \log_8(x+4)$

[10] 3. For $f(x) = 18 - 3x$ and $g(x) = 4^{x-4}$ find the following:

(A) $\sum_{k=1}^{60} f(k) = f(1) + f(2) + f(3) + \cdots + f(60)$.

(B) $\sum_{h=0}^{30} g(h) = g(0) + g(1) + g(2) + \cdots + g(30)$.

[10] 4. Beginning in January, a person plans to deposit \$100 at the end of each month into an account earning 6% compounded monthly. Each year taxes must be paid on the interest earned during that year. Find the interest earned during each year for the first 3 years.

[10] 5. A family has a \$129,000, 20-year mortgage at 7.2% compounded monthly.

(A) Find the monthly payment and the total interest paid during the entire life of the mortgage.

(B) Suppose the family decides to add an extra \$102.41 to its mortgage payment each month starting with the very first payment. How long will it take the family to pay off the mortgage?

(C) How much interest will be saved after adding the extra \$102.41?

[10] 6. Solve by using Gauss-Jordan Elimination:

$$\begin{aligned} 3x_1 + 8x_2 - x_3 &= -18 \\ 2x_1 + x_2 + 5x_3 &= 8 \\ 2x_1 + 4x_2 + 2x_3 &= -4 \end{aligned}$$

<u>No other method of solving these systems of equations will be accepted!</u>

[10] **7.** An economy is based on three sectors, agriculture, manufacturing, and energy. Production of a dollar's worth of agriculture requires an input of \$0.20 from agriculture, \$0.20 from manufacturing, and \$0.20 from energy. Production of a dollar's worth of manufacturing requires an input of \$0.40 from agriculture, \$0.10 from manufacturing, and \$0.10 from energy. Production of a dollar's worth of energy requires an input of \$0.30 from agriculture, \$0.10 from manufacturing, and \$0.10 from energy.

(A) Write the technological matrix M for this economy.

(B) If a final demand of \$10 billion for agriculture, \$20 billion for manufacturing, and \$15 billion for energy is to be met, then set up the equation to be satisfied by the inputs from the respective sectors.

(C) Solve the respective inputs satisfying these demands.

[10] **8.** Extremize $P(x,y) = 60x + 20y$ subject to

$$2x + 2y \geq 4,\ 6x + 4y \leq 36,\ 2x + y \leq 10,\ x \geq 0,\ y \geq 0\,.$$

[10] **9.** A county park system rates its 20 golf courses in increasing order of difficulty as bronze, silver, or gold. There are only two gold courses and twice as many bronze as silver courses.

(A) If a golfer decides to play a round at a silver or gold course, how many selections are possible?

(B) If a golfer decides to play one round per week for 3 weeks, first on a bronze course, then silver, then gold, how many combined selections are possible?

[10] **10.** If each of 5 people is asked to identify his or her favorite book from a list of 10 best-sellers, what is the probability that at least 2 of them identify the same book?

1. (A) Revenue is max at its vertex (since it is a parabola) x-coordinate of the vertex is in the middle between x-intercepts: $R(x)=0$ if

$x=0$ or $60x = 2{,}000 \Rightarrow x = \frac{100}{3}$

So the max of $R(x)$ is at $x = \frac{50}{3}$

$p\left(\frac{50}{3}\right) = 2{,}000 - 60 \cdot \frac{50}{3} = \$1{,}000$ per computer

(B) $R(x) = C(x)$

$2{,}000x - 60x^2 = 4{,}000 + 500x$

$\Rightarrow \quad 60x^2 - 1{,}500x + 4{,}000 = 0 \quad / \div 20$

$3x^2 - 75x + 200 = 0$

$$x = \frac{75 \pm \sqrt{(75)^2 - (4)(3)\cdot 200}}{6} = \begin{cases} 3.035 \\ 21.965 \end{cases}$$

Break-even points:

$(3.035, 5{,}517.576)$ & $(21.965, 14{,}982.424)$

in thousands

or (3035, \$5517576) &
(21965, \$14982424)

(C) Loss will occur for the values of $x \in [1, 3.035) \cup (21.965, 25]$ where 1 unit = 1,000 computers or on $[1{,}000;\ 3035) \cup (21965, 25000]$

Profit will occur on the interval:

$(3.035, 21.965)$ or $(3035, 21965)$
if 1 unit $\overset{?}{=}$ 1,000 computers | 1 unit = 1 computer

2. (A) $e^{12} = (e^4)^x e^{x^2} \Leftrightarrow e^{12} = e^{4x+x^2}$

$\Rightarrow 12 = 4x + x^2$ or $x^2 + 4x - 12 = 0$

factors: 6 & -2 $\quad (x+6)(x-2) = 0$

Hence $\underline{x=-6}$ or $\underline{x=2}$

2(B) $27 = 3^3$ so $(3^3)^{2x} = 3^{x^2-7}$ ③

$\Rightarrow$ $6x = x^2 - 7$ or $x^2 - 6x - 7 = 0$

factors: -7 & 1 $(x-7)(x+1) = 0$ ✓

$\Rightarrow$ $\underline{x = 7}$ or $\underline{x = -1}$

(C) $\log_b x = \log_b (4)^{3/2} - \log_b (8)^{2/3} + \log_b (2)^2$

$4^{3/2} = (\sqrt{4})^3 = 8$; $8^{2/3} = (\sqrt[3]{8})^2 = 4$

So $x = \dfrac{8 \cdot 4}{4} = \underline{\underline{8}}$

(D) $\log_a x(x-4) = \log_a 21$

$x^2 - 4x = 21 \Rightarrow x^2 - 4x - 21 = 0$

$(x-7)(x+3) = 0$

$\Rightarrow$ $\underline{\underline{x = 7}}$ or $\cancel{x = -3}$ not in the domain of log.

2 (E) $\log_8(x+6) + \log_8(x+4) = 1$

$\log_8 (x+6)(x+4) = 1$ $\qquad 1 = \log_8 8^1$

$(x+6)(x+4) = 8$

$x^2 + 10x + 24 - 8 = 0 \Rightarrow x^2 + 10x + 16 = 0$

$(x+8)(x+2) = 0$

$\Rightarrow x = -8$ or $x = -2$

Check the domain:

$x + 6 > 0$ & $x + 4 > 0$

$x > -6$ & $x > -4$ so

the only solution is $\underline{\underline{x = -2}}$

3. (A) Arithmetic sequence

$S_{60} = 60 \frac{f(1) + f(60)}{2} = 30[15 + (-162)] =$

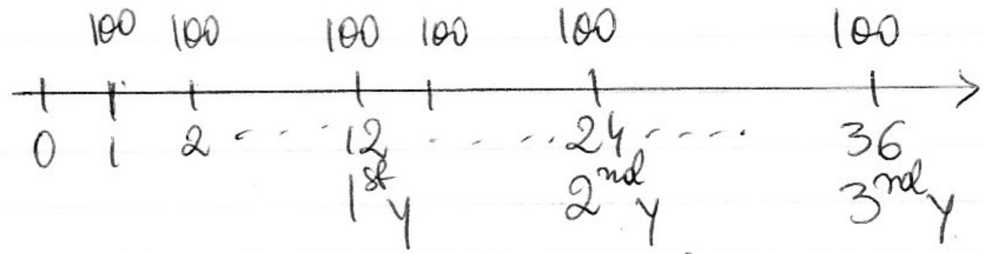
$= \underline{-4{,}410}$

(B) Geometric: $S_{31} = g(0) \frac{r^{31} - 1}{r - 1} = \frac{1}{256} \frac{4^{31} - 1}{3}$

$r = 4 \quad g(0) = 4^{-4} = \frac{1}{256}$

$= \underline{\underline{6.0048 \cdot 10^{15}}}$

(4)



Interest earned in the 1^{st} y:

$$I_1 = FV_{1^{st} y} - \underbrace{12 \cdot 100}_{\text{deposits}} = \underline{\$33.56}$$

$$FV_{1^{st}} = 100 \, \frac{(1.005)^{12} - 1}{0.005} = 1{,}233.56$$

Interest earned in the 2^{nd} year:

$$I_2 = FV_2 - FV_1 - 12 \cdot 100 \overset{or}{=}$$

$$= FV_2 - 24 \cdot 100 - I_1 = \underline{\$109.64}$$

$$FV_2 = 100 \, \frac{(1.005)^{24} - 1}{0.005} = 2{,}543.20$$

Interest earned in the 3^{rd} year:

$$I_3 = FV_3 - FV_2 - 12 \cdot 100 \overset{or}{=}$$

$$= FV_3 - 36 \cdot 100 - I_1 - I_2 = \underline{\$190.41}$$

$$FV_3 = 100 \, \frac{(1.005)^{36} - 1}{0.005} = 3{,}933.61$$

5. Loan = PV = 129,000

$i = \frac{0.072}{12} = 0.006 \qquad n = 20 \cdot 12 = 240$

(A) $129{,}000 = PMT \, \frac{1-(1.006)^{-240}}{0.006}$

$\Rightarrow$ PMT = $\underline{\$1{,}015.68}$

Total interest paid: $240 \cdot PMT - PV = \underline{\underline{114{,}763.2}}$

(B) $PMT^{new} = 1015.68 + 102.41 = 1{,}118.09$

$129{,}000 = 1{,}118.09 \, \frac{1-(1.006)^{-n}}{0.006}$

$0.692251965 = 1-(1.006)^{-n}$

$\Rightarrow (1.006)^{-n} = 1 - 0.6922519\ldots$

$(-n) \ln 1.006 = \ln[\quad]$

$\Rightarrow n = -\frac{\ln 0.30774803 5}{\ln 1.006} = 197$ months

i.e. 16 years & 5 months

(C) $240 \cdot PMT - 197 \cdot PMT^{new} =$

$= 243{,}763.20 - 220{,}263.73 = \underline{\underline{23{,}499.47}}$

6. Augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 8 & -1 & -18 \\ 2 & 1 & 5 & 8 \\ 2 & 4 & 2 & -4 \end{array}\right] \xrightarrow[R_3^n = R_3 - R_2]{R_1^n = R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 7 & -6 & -26 \\ 2 & 1 & 5 & 8 \\ 0 & 3 & -3 & -12 \end{array}\right] \xrightarrow[R_3^n = \frac{1}{3} R_3]{R_2^n = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 7 & -6 & -26 \\ 0 & -13 & 17 & 60 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$-2R_1 \left[\begin{array}{ccc|c} -2 & -14 & 12 & 52 \end{array}\right]$$

$$-7R_3 \left[\begin{array}{ccc|c} 0 & -7 & +7 & +28 \end{array}\right]$$

$$13R_3 \left[\begin{array}{ccc|c} 0 & 13 & -13 & -52 \end{array}\right]$$

$$\xrightarrow[R_2^n = R_2 + 13R_3]{R_1^n = R_1 - 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \\ 0 & 1 & -1 & -4 \end{array}\right] \xrightarrow{R_2^n = \frac{1}{4} R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right] \xrightarrow[R_3^n = R_3 + R_2]{R_1^n = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So $x_1 = 0$, $x_2 = -2$ & $x_3 = 2$ is the solution of this system

7. (A)

$$M = \begin{array}{c} \\ A \\ M \\ E \end{array} \begin{array}{c} \begin{array}{ccc} A & M & E \end{array} \\ \left[\begin{array}{ccc} 0.2 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \end{array}\right] \end{array}$$

(B) $D = \begin{bmatrix} 10 \\ 20 \\ 15 \end{bmatrix}$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{array}{c} A \\ M \\ E \end{array}$ in billions \$

$$X = MX + D \quad \text{or} \quad (I - M)X = D$$

(C) $[I - M \mid D] = \left[\begin{array}{ccc|c} 0.8 & -0.4 & -0.3 & 10 \\ -0.2 & 0.9 & -0.1 & 20 \\ -0.2 & -0.1 & 0.9 & 15 \end{array}\right] \xrightarrow{10 \cdot R_i}$

$$\left[\begin{array}{ccc|c} 8 & -4 & -3 & 100 \\ -2 & 9 & -1 & 200 \\ -2 & -1 & 9 & 150 \end{array}\right] \xrightarrow[R_2^n = R_2 - R_3]{R_1^n = R_1 + 4R_3} \left[\begin{array}{ccc|c} 0 & -8 & 33 & 700 \\ 0 & 10 & -10 & 50 \\ -2 & -1 & 9 & 150 \end{array}\right]$$

$4R_3 [-8 \ -4 \ 36 \mid 600]$

$$\xrightarrow[R_3 \Leftrightarrow R_1]{R_2^n = \frac{1}{10} R_2} \left[\begin{array}{ccc|c} -2 & -1 & 9 & 150 \\ 0 & 1 & -1 & 5 \\ 0 & -8 & 33 & 700 \end{array}\right] \xrightarrow[R_3^n = R_1 + 8R_2]{R_1^n = R_1 + R_2}$$

$$\rightarrow \left[\begin{array}{ccc|c} -2 & 0 & 8 & 155 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 25 & 740 \end{array}\right] \xrightarrow[R_1^n = -R_1]{R_3^n = \frac{1}{25} R_3} \left[\begin{array}{ccc|c} 2 & 0 & -8 & -155 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & 29.6 \end{array}\right]$$

9

7. (C) $\xrightarrow[R_2^n = R_1 + R_3]{R_1^n = R_1 + 8R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 81.8 \\ 0 & 1 & 0 & 34.6 \\ 0 & 0 & 1 & 29.6 \end{array}\right]$

$\xrightarrow{R_1^n = \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40.9 \\ 0 & 1 & 0 & 34.6 \\ 0 & 0 & 1 & 29.6 \end{array}\right]$

They need to produce \$40.9 billion from Agriculture & \$34.6 billion from Manufacturing & \$29.6 — " — from Energy to satisfy the final demands.

8. $\begin{cases} 2x+2y \geq 4 & L_1 \\ 6x+4y \leq 36 & L_2 \\ 2x+y \leq 10 & L_3 \\ x, y \geq 0 \end{cases}$

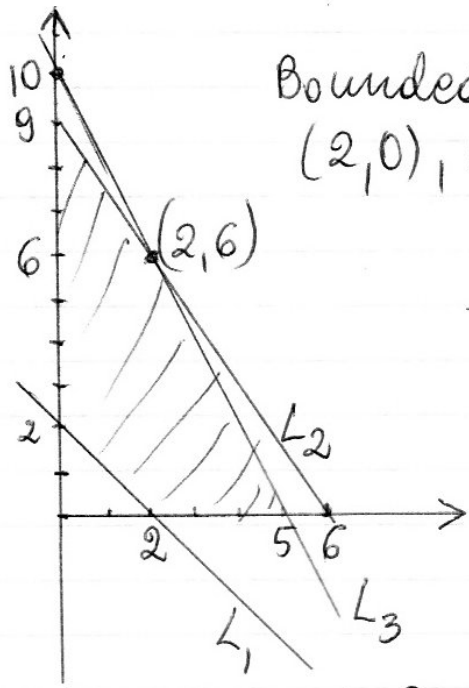
L_1: $x+y=2$
$(2,0)$, $(0,2)$

L_2: $3x+2y=18$
$x=0 \Rightarrow y=9$
$y=0 \Rightarrow x=6$

L_3: $2x+y=10$
$(0,10)$ & $(5,0)$

$L_2 \cap L_3$: $\begin{cases} 3x+2y=18 & /\cdot(-1) \\ 2x+y=10 & /\cdot 2 \end{cases}$

$-3x-2y=-18$
$4x+2y=20$
sum: $x=2$ & $y=6$



Bounded region with corner points:
$(2,0)$, $(5,0)$, $(2,6)$, $(0,9)$ & $(0,2)$

Corner Point	$P = 60x + 20y$
$(2,0)$	120
$(5,0)$	300 ⇐ max
$(2,6)$	$120 + 120 = 240$
$(0,9)$	180
$(0,2)$	40 ⇐ min.

The max. of $P = 300$ at $(5,0)$ & the min. is 40 at $(0,2)$

9. There are 2-Gold, x-Silver & 2x-Bronze

$2 + x + 2x = 20 \Rightarrow 3x = 18 \Rightarrow x = 6$

$G \Leftrightarrow 2$ $\quad S \Leftrightarrow 6$ $\quad B \Leftrightarrow 12$ courses

(A) $G + S = 8$ courses so 8 selections are possible $C_{8,1} = \binom{8}{1} = \underline{\underline{8}}$

(B) $C_{12,1} \cdot C_{6,1} \cdot C_{2,1} = 12 \cdot 6 \cdot 2 = \underline{\underline{144}}$

10. $A = \{$at least two choose the same book$\}$

$A' = \{$all 5 choose different book$\}$ $\quad n(A') = P_{10,5}$

$n(S) = 10 \cdot 10 \cdot \ldots \cdot 10 = 10^5$

$Pr(A) = 1 - Pr(A') = 1 - \dfrac{10!}{5!\, 10^5}$