MAT 1702C, Fall 2012

Assignment 4

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Due December 4, 2012

For full marks show all details of your work!

1. **(2 points)** Let $z = \frac{2+2i}{1-i}$.

   - Write z in the form $a + ib$.

Solution:

- We multiply both numerator and denominator of z by the conjugate of $1 - i$, namely $1 + i$. We have

$$z = \frac{(2+2i)(1+i)}{(1-i)(1+i)} = \frac{2(1+i)^2}{|1-i|^2} = \frac{4i}{2} = 2i\,.$$

2. Consider the following matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) **(1.5 point)** Find the characteristic polynomial of A.

Solution: Note that $P_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$. Expanding across Row 3 one gets that $P_A(\lambda) = (-1)^{3+3}(-1-\lambda)\det \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix}$. Using the definition of the determinant of a 2×2 matrix, one has that $P_A(\lambda) = (-1-\lambda)(1-\lambda)^2 = -(1-\lambda)^2(1+\lambda)$.

(b) **(2 points)** List the eigenvalues of A with their multiplicities.

Solution: We use part a). Solving $P_A(\lambda) = -(1-\lambda)^2(1+\lambda) = 0$, one gets that $\lambda_1 = 1$ and $\lambda_2 = -1$ are the eigenvalues. The multiplicity of $\lambda_1 = 1$ is 2, and the multiplicity of $\lambda_2 = -1$ is 1.

c) **(2 points)** For the eigenvalues found in part b) describe the eigenvectors.

Solution: We should find $Nul(A - \lambda I_3)$ for each of the above eigenvalues.
For $\lambda_1 = 1$ note that

$$A - \lambda_1 I_3 = A - I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence in the homogeneous linear system $(A - I_3)\mathbf{x} = \mathbf{0}$ the variable x_2 is free, and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

A basis for the eigenspace associated to $\lambda_1 = 1$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_2 = -1$ note that

$$A - \lambda_2 I_3 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{-1}{4} R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore in the linear system $(A + I_3)\mathbf{x} = \mathbf{0}$ the variable x_3 is a free variable, and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2x_3 \\ 1/4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace associated to $\lambda_2 = -1$ is $\left\{ \begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix} \right\}$.

d) **(1.5 points)** Is the matrix A diagonalizable? Justify your answer.

Solution: The dimension of the eigenspace associated with $\lambda = 1$ is $1 < 2$ the multiplicity of $\lambda = 1$. By a result given in class this matrix is not diagonalizable.

3. The population of rats in my farm moves constantly back and forth between the house and the field (what a life!). Every week 40% of the rats in the field will go to my house while 30% of those in my house move to the field. Suppose that, initially, the distribution of this population is: 70% in the field and 30% in my house.

(a) **(1 point)** Find the migration matrix and set up a difference equation for this situation.
(b) **(1 point)**What is the rats' distribution after two week in my farm?

Solution: (a) If $\mathbf{x}_k = \begin{bmatrix} a_k \\ c_k \end{bmatrix}$ then

$$M = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

$\mathbf{x}_{k+1} = M\mathbf{x}_k$, where $k = 0, 1, 2, \ldots$

Solution: (b) Using (a), we have

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.51 \\ 0.49 \end{bmatrix}$$

and

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.51 \\ 0.49 \end{bmatrix} = \begin{bmatrix} 0.453 \\ 0.547 \end{bmatrix}.$$