MAT1322 – Notes — By Dr. Hua

Contents

Chapter 5 – Integrals

5.10 Improper Integrals

Type I: Infinite Intervals

$$\begin{aligned}\int_a^\infty f(x)dx &= \lim_{t\to\infty}\int_a^t f(x)dx, \quad \int_{-\infty}^b f(x)dx = \lim_{t\to\infty}\int_t^b f(x)dx,\\ \int_{-\infty}^\infty f(x)dx &= \int_c^\infty f(x)dx + \int_{-\infty}^c f(x)dx.\end{aligned}$$

Definition: Integral is convergent (or divergent) $\Leftrightarrow$ Integral is a finite number (or ∞).

Example 1

$$\int_1^\infty \frac{1}{x^2}dx = 1.$$

Example 2

$$\int_1^\infty \frac{1}{x}dx = \infty.$$

Example 3

$$\int_{-\infty}^1 \frac{1}{1+x^2}dx = \frac{3\pi}{4}.$$

Example 4

$$\int_{-\infty}^\infty e^{-|x|}dx = 2.$$

Example 5 *Determine if the integral*

$$I = \int_2^\infty xe^{-x}dx$$

is convergent or divergent and evaluate if it is convergent.

Solution. $I = \lim_{t\to\infty} \int_2^t xe^{-x}\, dx$ (integration by parts: let $u = x$ and $dv = e^{-x}dx$)

$= \lim_{t\to\infty} \left(x(-e^{-x})|_2^t - \int_2^t -e^{-x}\, dx \right)$

$= \lim_{t\to\infty} (-xe^{-x} - e^{-x})|_2^t$

$= \lim_{t\to\infty} -(x+1)e^{-x}|_2^t$

$= \lim_{t\to\infty} (-(t+1)e^{-t} + 3e^{-2})$

$= 3e^{-2}$ (where $\lim_{t\to\infty}(t+1)e^{-t} = 0$ by L'Hospital's Rule).

Type 2: Discontinuous Integrands

If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)dx = \lim_{t\to b} \int_a^t f(x)dx;$$

If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)dx = \lim_{t\to a} \int_a^b f(x)dx;$$

If $f(x)$ is **discontinuous** at c: $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 6

$$\int_2^3 \frac{1}{\sqrt{3-x}}dx = \lim_{t\to 3} \int_2^t \frac{1}{\sqrt{3-x}}dx = \lim_{t\to 3}[-2\sqrt{3-x}]_2^t = 2.$$

Example 7 *Determine if the integral*

$$\int_0^2 \frac{1}{x-1}dx$$

is convergent or divergent and evaluate if it is convergent.

Example 8 $\int_0^e \ln x dx = 0.$

p-Integral

Example 9

$$\int_1^\infty \frac{1}{x^p}dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1; \\ \text{divergent}, & \text{if } p \leq 1. \end{cases}$$

Example 10

$$\int_0^1 \frac{1}{x^p}dx = \begin{cases} \frac{1}{1-p}, & \text{if } p < 1; \\ \text{divergent}, & \text{if } p \geq 1. \end{cases}$$

Comparison Test for Improper Integral

If $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x) \geq 0$ on $x \geq a$. Then

(i) $\int_a^\infty f(x)dx$ is convergent $\Longrightarrow \int_a^\infty g(x)dx$ is convergent;

(ii) $\int_a^\infty g(x)dx$ is divergent $\Longrightarrow \int_a^\infty f(x)dx$ is divergent.

Example 11

$$\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx = \text{convergent}.$$

$\because$

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{x^{3/2}}.$$

Example 12

$$\int_8^\infty \frac{1+\sqrt{x}}{x-6}dx = \text{divergent}.$$

$\because$

$$\frac{1+\sqrt{x}}{x-6} \geq \frac{1}{\sqrt{x}}.$$

Example 13 *Determine whether the integral*

$$\int_2^\infty \frac{\cos^4 x}{e^x + \sin^2 x + 1}dx$$

is convergent or divergent.

Solution: Since

$$\frac{\cos^4 x}{e^x + \sin^2 x + 1} \leq \frac{1}{e^x} = e^{-x},$$

$$\int_2^\infty \frac{\cos^4 x}{e^x + \sin^2 x + 1}dx \leq \int_2^\infty e^{-x}dx = e^{-2}.$$

By Comparison Theorem, the original integral is convergent.

Example 14 *Determine whether the integral*

$$\int_0^1 \frac{1}{x^{1.9}\sin^2 x}dx$$

is convergent or divergent.

Solution: Let $y = 1/x$. Then

$$\int_0^1 \frac{1}{x^{1.9}\sin^2 x}dx = \int_1^\infty \frac{1}{y^{0.1}\sin^2 \frac{1}{y}}dy \geq \int_1^\infty \frac{1}{y^{0.1}}dy = \infty.$$

Chapter 6 – Applications of Integration

6.1 More about Areas

Theorem. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then the area of the region bounded by

$$y = f(x), y = g(x), x = a, x = b$$

is

$$A = \int_a^b (f(x) - g(x))dx.$$

Remark. To find the area A of the region enclosed by $y = f(x)$ and $y = g(x)$: Find the intersections, then use the theorem.

Example. Calculate the area of the region bounded by

$$y = x^2 - 4x + 7, y = -x^2 + 4x + 1, x = 0, x = 2.$$

Sol: Step 1. Find intersections: Let $(-x^2 + 4x + 1) = (x^2 - 4x + 7), \Rightarrow x^2 - 4x + 3 = 0, \Rightarrow x = 1, x = 3$.

Step 2. By using the intersections, the interval $(0, 2)$ is divided into $(0, 1)$ and $(1, 2)$. In $(0, 1)$: $x^2 - 4x + 7 > -x^2 + 4x + 1$; In $(1, 2)$: $-x^2 + 4x + 1 > x^2 - 4x + 7$. Therefore

$$A = \int_0^1 [(x^2 - 4x + 7) - (-x^2 + 4x + 1)]dx + \int_1^2 [(-x^2 + 4x + 1) - (x^2 - 4x + 7)]dx$$

$$= \int_0^1 (2x^2 - 8x + 6)dx + \int_1^2 (-2x^2 + 8x - 6)dx.$$

Example. Find the area of the region between $y = x^{1/2}$ and $y = x^{1/3}$ for $0 \leq x \leq 1$.

Example. Find the area of the region bounded by the parabolas $y = 2x - x^2$ and $y = x^2$.

Sol: Step 1. Find intersections: $2x - x^2 = x^2 \Rightarrow x = 0, x = 1$.

Step 2. For $0 < x < 1$, $2x - x^2 > x^2$.

$$A = \int_0^1 [2x - x^2 - x^2]dx = 1 - \frac{2}{3} = \frac{1}{3}.$$

Example. Calculate the area of the region bounded by

$$y = \sin x, y = \cos x, x = 0, x = \frac{\pi}{2}.$$

Sol: $A = 2\sqrt{2} - 2$.

Theorem. If the region is bounded by

$$x = f(y), x = g(y), y = c, y = d$$

where $f(y) \geq g(y)$ for $c \leq y \leq d$, then the area of the region is

$$A = \int_c^d (f(y) - g(y))dy.$$

Example. Calculate the area of the region enclosed by

$$y = x - 1, y^2 = 2x + 6.$$

Sol:

$$A = \int_{-2}^4 [(y-1) - (\frac{1}{2}y^2 - 3)]dy = 18.$$

6.2 Volume

Definition 1 *Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is:*

$$V = \int_a^b A(x)dx.$$

Example. Find the volume of a sphere of radius r.

Solution: $A(x) = \pi(r^2 - x^2)$.

Example. Find the volume of a pyramid whose base is a square with side L and whose height is h.

Volumes of revolution

Case 1: If we rotate a region bounded by $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ **around the line** $y = k$ **(**$k = 0 \Longleftrightarrow$ **x-axis)**, where $f(x) \geq g(x) \geq 0$, then

$$A(x) = \pi[f(x) - k]^2 - \pi[g(x) - k]^2, \quad V = \int_a^b A(x)dx.$$

Case 2: If we rotate a region bounded by $x = f(y)$, $x = g(y)$, $y = c$, $y = d$ **around the line** $x = k$ **(**$k = 0 \Longleftrightarrow$ **y-axis)**, where $f(y) \geq g(y) \geq 0$, then

$$A(y) = \pi[f(y) - k]^2 - \pi[g(y) - k]^2, \quad V = \int_a^b A(y)dy.$$

Example. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Sol: $A(x) = \pi(\sqrt{x})^2$.

Example. Find the volume of the solid obtained when the region bounded by $y = x^2$, $y = 0$,
$x = 1$ and $x = 2$ is rotated around the line $y = -3$.

Sol: The cross-sectional area at x is:

$$A(x) = \pi R^2 - \pi r^2, \quad R = x^2 - (-3) = x^2 + 3, r = 0 - (-3) = 3.$$

$$\Rightarrow A(x) = \pi(3 + x^2)^2 - \pi 3^2 = \pi(x^4 + 6x^2).$$

$$V = \int_1^2 A(x)dx = \int_1^2 \pi(x^4 + 6x^2)dx = 20.2\pi.$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and x = 0 about the y-axis.

Sol: $A(y) = \pi x^2 = \pi y^{2/3}$.

6.3 Volumes by Cylindrical Shells

(1) If we rotate a region bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$ about the y-axis, then

$$V = 2\pi \int_a^b x f(x) dx.$$

(2) If we rotate a region bounded by $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ (where $f(x) \geq g(x)$ for $x \in (a, b)$) about the y-axis, then

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

(3) If we rotate a region bounded by $x = f(y)$, $x = g(y)$, $y = c$, $y = d$ (where $f(y) \geq g(y)$ for $y \in (c, d)$) about the x-axis, then

$$V = 2\pi \int_c^d y[f(y) - g(y)]dy.$$

Example. Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and x = 0 about the y-axis.

Sol: When $y = 8$, $x = 2$. Thus

$$V = 2\pi \int_0^2 x[8 - x^3]dx = 19.2\pi.$$

Example. Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $y = 0$, and $x = 8$ about the x-axis.

Sol: When $x = 8$, $y = 2$. From $y = \sqrt[3]{x}$ we have $x = y^3$. Thus

$$V = 2\pi \int_0^2 y[8 - y^3]dy = 19.2\pi.$$

Example. Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 4$.

Sol: The intersection is $x = 0, 1$. The radius of the cylindrical shell at x is $4 - x$, the circumference is $2\pi(4 - x)$, and the height is $y = x - x^2$. Thus

$$V = \int_0^1 2\pi(4 - x)(x - x^2)dx.$$

Example. Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 3x - x^2$ and $y = 2x$ about the line $x = 4$.

Sol: The intersection is $x = 0, 1$. The radius of the cylindrical shell at x is $4 - x$, the circumference is $2\pi(4 - x)$, and the height is $y = x - x^2$. Thus

$$V = \int_0^1 2\pi(4 - x)(x - x^2)dx.$$

6.4 Arc Length

The length of the arc given by the parametric equations $x = x(t), y = y(t)$ $(a \le t \le b)$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

There are two special cases $t = x$ or $t = y$:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy.$$

Example. Calculate the length of the arc $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 0$ and $x = 2$.

Solution.

$$L = \int_0^2 \sqrt{1 + [f'(x)]^2}dx = \int_0^2 \sqrt{1 + [\frac{1}{2}(e^x - e^{-x})]^2}dx = \int_0^2 \sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}dx$$

$$= \int_0^2 \sqrt{\frac{1}{4}(e^{2x} + 2 + e^{-2x})}dx = \int_0^2 \frac{1}{2}(e^x + e^{-x})dx = \frac{1}{2}(e^x - e^{-x})|_0^2$$

$$= \frac{1}{2}(e^2 - e^{-2})$$

Example. Calculate the length of the arc

$$x(t) = t^2 + 10, \ y(t) = \frac{2}{3}t^3, \quad 0 \le t \le 3.$$

$x'(t) = 2t$, $y'(t) = 2t^2$. Hence

$$L = \int_0^3 \sqrt{x'(t)^2 + y'(t)^2}dt = \int_0^3 \sqrt{(2t)^2 + (2t^2)^2}dt$$

$$= \int_0^3 2t\sqrt{1 + t^2}dt = \int_0^3 \sqrt{1 + t^2}d(1 + t^2)$$

$$= \frac{2}{3}(1+t^2)^{3/2}|_0^3 = \frac{2}{3}[(10)^{3/2} - 1] = 20.41.$$

Example. Calculate the length of the arc of one arch of the cycloid

$$x(t) = r(t - \sin t), \;\; y(t) = r(1 - \cos t).$$

Solution: Let $y = 0$, we get $t = 2\pi, \ldots$. Thus one arch is for $0 \le t \le 2\pi$. Hence

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} dt = r \int_0^{2\pi} \sqrt{2(1 - \cos t)} dt$$

$$= r \int_0^{2\pi} \sqrt{4 \sin^2(t/2)} dt == r \int_0^{2\pi} 2|\sin t/2| dt = 8r.$$

6.5 Average Value of a Function

. Definition:

$$f_{ave}[a,b] = \frac{1}{b-a} \int_a^b f(x) dx.$$

MEAN VALUE THEOREM for Integrals: Let f be continuous on the closed interval [a, b], then there exists $c \in [a,b]$ such that

$$f(c) = f_{ave}[a,b].$$

Example. Let $f(x) = 1 + 4x^3$. Find $c \in [0,1]$ such that $f(c) = f_{ave}[0,1]$.

Example. Let $f(x) = \ln x$. Find $f_{ave}[e, 3e]$.

6.6 Applications to Physics and Engineering

Work done

- If force is a constant, then: **work done= force** × **distance,**

- If force is a function $f(x)$, then the work done in moving an object from a to b is:

$$W = \int_a^b f(x)\, dx.$$

force (in N) = mass (in lb or kg) = mass in kg × acceleration in m/s^2

Work done can have unit ft-lb, or N-m (Joules).

1 ft-lb = 1.36 J, 1lb = $\frac{9.8}{2.2}$ N = 4.45 N, 1ft=0.305m.

Example. When a particle is located a distance x feet from the origin, a force of $3x^2 + 4x$ pounds acts on it. How much work is done in moving it from $x = 0$ to $x = 1$?

Solution:

$$W = \int_0^1 (3x^2 + 4x)dx = 3(ft - lb).$$

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is given by: $f(x) = kx$, where $k > 0$ is called the spring constant.

The work done to stretch a spring with natural length a meters and force f N from b meters to c meters is:

$$W = \int_{b-a}^{c-a} kxdx,$$

where f N is the force required to hold the spring stretched from its natural length to b meters.

Example. A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm to a length of 15cm. How much work is done in stretching the spring from 15cm to 18cm?

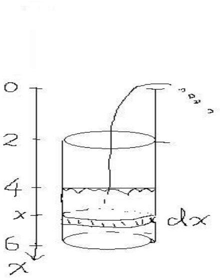
Solution: 15-10=5cm=0.05m. $f(0.05) = 40, \Rightarrow 0.05k = 40, \Rightarrow k = 800$. Thus $f(x) = 800x$.

$$W = \int_{0.05}^{0.08} 800xdx = 1.56J.$$

Example. A cylindrical tank of radius 2 m and height 4 m is half-full of water. Find the work (J) required to pump all of the water to a point that is 2 m above the top of the tank. The density of water is $\rho = 1000$ kg/m^3 and the acceleration of gravity is $g = 9.8$ m/s^2.

Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.

Sol: We introduce the axis as in the graph.



Take a slice at h with height Δh. The work done to the slice is:

$$\Delta W = \rho \Delta V g h = \rho \pi 2^2 \Delta h g h = 4\pi \rho g h \Delta h.$$

Hence

$$W = \int_{2+4/2}^{2+4} \Delta W = \int_4^6 4\pi \rho g h dh = 40\pi \rho g = 392000\pi \doteq 1.23 \times 10^6 J.$$

Pressure

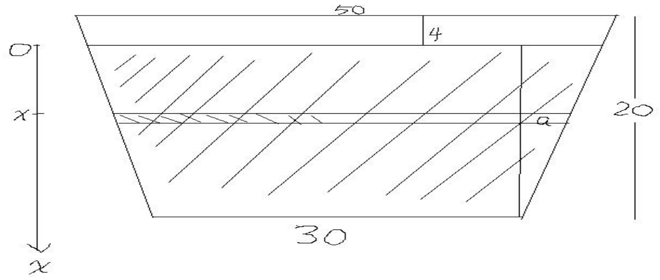
Pressure (force per unit area) $= \rho g h =$ density$\cdot g\cdot$depth, force = pressure$\cdot$area.

Example. A regular trapezoid plate with top side 50m, bottom side 30m, height 20m, is pushed into water such that top part is outside with height 4m. Find the force on the plate due to pressure.

Solution. We introduce x-axis downward with origin water level.

Then at the position x, the length on the plate is $(30 + 2a)$, where

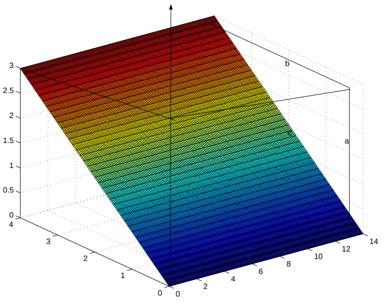
$$\frac{a}{10} = \frac{16-x}{20}, a = \frac{16-x}{2}.$$



Thus $\Delta A = [30 + 2a]dx = (46 - x)dx$, and

$$f = \int_0^{16} \rho g x \Delta A = \int_0^{16} \rho g x (46 - x) dx \approx 4.43 \times 10^7 (N).$$

Example. A trough of length 14m with the triangular cross-section (where three sides are $a = 3m$, $b = 4m$ and $c = 5m$, side b is on the top) is filled with water. Compute the hydrostatic force on each of its sides due to the pressure of the water.



Solution: (1) Force on side a: $F_a = \int_0^3 1000 g x 14 dx$.

(2) Force on side c: By using similar triangle, $\Delta c = \frac{5}{3}\Delta a$, thus $F_c = \int_0^3 1000 g x 14 (\frac{5}{3}) dx$.

(3) Force on the end: By using similar triangle, the length of a small piece is $\frac{4}{3}(3 - x)$ when the depth is x, thus $F_{end} = \int_0^3 1000 g x \frac{4}{3}(3 - x) dx = 58800 (N)$.

Density and Center of Mass

- The center of mass $\bar{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is:

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx},$$

where $\delta(x)$ is the density of the object. The total mass is:

$$m = \int_a^b \delta(x)dx.$$

- For a plate bounded by $y = f(x)$ and $y = g(x)$ with $a \le x \le b$ and $f(x) \ge g(x)$, where the density ρ is a constant. The center $(\bar{x}, \bar{y})$ is:

$$\bar{x} = \frac{\int_a^b x(f-g)dx}{\int_a^b (f-g)dx}, \quad \bar{y} = \frac{\int_a^b \frac{1}{2}(f^2 - g^2)dx}{\int_a^b (f-g)dx}.$$

The total mass:

$$m = \rho \int_a^b (f-g)dx,$$

where

$$M_y = \int_a^b \rho x(f-g)dx, M_x = \int_a^b \rho\frac{1}{2}(f^2 - g^2)dx$$

are the moments about y and x respectively.

Example. Find the center of mass $\bar{x}$ of a rod lying along the x-axis between $x = 0$ and $x = 1$ with density $\delta(x) = 1 + \sin x$.

Example. A rod is placed on x-axis from $x = a$ to $x = b$ with density $\delta(x)$. Find the center of mass $\bar{x}$ if $a = 0$, $b = 1$, $\delta(x) = e^{2x}$.

Sol:

$$\bar{x} = \frac{\int_0^1 xe^{2x}dx}{\int_0^1 e^{2x}dx} = \frac{[\frac{1}{2}xe^{2x} - \frac{1}{2}e^{2x}]|_0^1}{[\frac{1}{2}e^{2x}]|_0^1} = \frac{1}{e^2 - 1}.$$

Example. A plate, with a constant mass density of $\rho = 1$, has the shape bounded by

$$y = x^2, \quad y = 0, \quad x = 0, \quad x = 2.$$

(a) Find the total mass M.

(b) Find the center of mass $(\bar{x}, \bar{y})$.

Sol: (a)

$$M = \rho \int_0^2 x^2 dx = \frac{8}{3}.$$

(b)

$$M_y = \rho \int_0^2 x(x^2) dx = \int_0^2 x^3 dx = 4,$$

$$M_x = \rho \int_0^2 \frac{1}{2}(x^2)^2 dx = \frac{1}{2} \int_0^2 x^4 dx = \frac{16}{5}.$$

$$\bar{x} = \frac{M_y}{M} = 1.5, \quad \bar{y} = \frac{M_x}{M} = 1.2.$$

$(\bar{x}, \bar{y}) = (1.5, 1.2)$.

Chapter 7 – Differential Equations

7.1 Modeling with differential equations

What is a differential equation?

First-order DE: $f(y', y, x) = 0$.

Second-order DE: $f(y'', y', y, x) = 0$.

n-th order DE: $f(y^{(n)}, ..., y', y, x) = 0$.

IVP: Initial value problem = DE with initial conditions.
Family of solutions: if a DE without initial condition, then it may have infinite solutions.

Example. Match the following DEs and possible solutions.

DE: (a) $y'' = y$ (b) $y' = -y$ (c) $y' = 1/y$ (d) $y'' = -y$ (e) $x^2y'' - 2y = 0$.

SOL: (1) $y = \cos x$ (2) $y = \cos(-x)$ (3) $y = x^2$ (4) $y = e^x + e^{-x}$ (5) $y = \sqrt{2x}$

7.2 Direction Fields and Euler's method

Direction Fields

Consider $y' = F(x, y)$. If we draw short line segments with slope $F(x, y)$ at points (x, y), the result is called direction fields (or slope fields).

Example. Sketch the direction field $y' = 2x + 3y$, $y(0) = -1$.

Euler's method

If h (or Δx) is the step size, $y' = F(x, y)$, $y(x_0) = y_0$, then

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}), \quad x_n = x_{n-1} + h.$$

Example. Use Euler's method with step size 0.1 to estimate $y(0.2)$, where y is the solution of the initial value problem $y' = 2x + y^2$, $y(0) = 1$.

Solution: Since

$$y(x+h) \approx y(x) + hy'(x) = y(x) + h\left(2x + y(x)^2\right),$$

we have

$$y(0.1) = 1 + 0.1(2(0) + 1^2) = 1.1,$$
$$y(0.2) = 1.1 + 0.1[2(0.1) + (1.1)^2] = 1.241.$$

Example. Use Euler's Method to solve the following differential equation (keep THREE decimal places):

$$\frac{df}{dt} = e^{-t}, \quad f(0) = 0.$$

(a) Use step size $\Delta t = 1$ to find $f(4)$.
(b) Use step size $\Delta t = 0.5$ to find $f(4)$.
(c) Which $f(4)$ is better, if $f(t) = 1 - e^{-t}$?

Solution: Solution: (a) Note that

$$f(t + \Delta t) = f(t) + f'(t)\Delta t = f(t) + e^{-t}.$$

t	Δt	$t + \Delta t$	$f(t + \Delta t) = f(t) + e^{-t}$
0	1	1	1
1	1	2	1.368
2	1	3	1.503
3	1	4	1.553

We obtain $f(4) = 1.553$.

(b) Note that

$$f(t + \Delta t) = f(t) + f'(t)\Delta t = f(t) + 0.5e^{-t}.$$

t	Δt	$t+\Delta t$	$f(t+\Delta t) = f(t) + 0.5e^{-t}$
0	0.5	0.5	0.5
0.5	0.5	1	0.803
1	0.5	1.5	0.987
1.5	0.5	2	1.099
2	0.5	2.5	1.167
2.5	0.5	3	1.208
3	0.5	3.5	1.233
3.5	0.5	4	1.248

We obtain $f(4) = 1.248$.

(c) Note that $f(4) = 1 - e^{-4} < 1$. So the answer in (b) is better.

We obtain $y(0.4) = 0.0036$.

Example. Consider the differential equation

$$\frac{dy}{dt} = ky, \quad y(0) = C.$$

Use Euler's method to approximate the solution $y(t)$ with step size $\Delta t = \frac{t}{n}$.

Solution: Let $t_0 = 0$, then $t_1 = \frac{t}{n}$, $t_2 = 2\frac{t}{n}$, ..., $t_n = t$. By Euler's method, we have

$$y(t+\Delta t) = y(t) + \Delta t y'(t) = y(t) + ky(t)\Delta t = y(t)\left(1 + k\Delta t\right),$$

that is

$$y(t_{j+1}) = y(t_j)\left(1 + k\Delta t\right).$$

Hence

$$y(t_1) = y(t_0)\left(1 + k\Delta t\right) = C\left(1 + k\Delta t\right);$$

$$y(t_2) = y(t_1)\left(1 + k\Delta t\right) = C\left(1 + k\Delta t\right)^2;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned} y(t_n) &= C\left(1 + k\Delta t\right)^n \\ &= C\left(1 + k\frac{t}{n}\right)^n. \end{aligned}$$

7.3 Separation of variables

Consider

$$\frac{dy}{dx} = g(x)f(y),$$

its solution is

$$\int \frac{1}{f(y)} dy = \int g(x) dx.$$

Example. Solve the following differential equation

$$\frac{dp}{dt} = (1+p^2)te^t, \quad p(0) = 1.$$

Solution: We write the equation as

$$\frac{dp}{1+p^2} = te^t dt,$$

$$\int \frac{dp}{1+p^2} = \int te^t dt.$$

Now we apply Integration-By-Parts to the right hand side. we obtain

$$\arctan p = te^t - e^t + c.$$

Since $p(0) = 1$, we have

$$\arctan 1 = 0 - 1 + C, \quad C = \pi/4 + 1.$$

Hence

$$\arctan p = te^t - e^t + \pi/4 + 1, \quad or \quad p = \tan(te^t - e^t + \pi/4 + 1).$$

Example. Solve the following differential equation

$$\frac{dy}{dt} - y^2 - y^2 t = 0, \quad y(1) = 2.$$

Example. Suppose we have the equation $\frac{dy}{dt} = \frac{4t}{3y^2 + \cos y}$.

(a) Solve the equation .

Solution: This is a separable equation. We rewrite the equation as

$(3y^2 + \cos y)dy = 4tdt$. Integrating it gives

$$y^3 + \sin y = 2t^2 + C,$$

where C is a constant.

(b) Find the solution satisfying the initial condition $y(\pi) = 0$.

From part a) we have

$$0^3 + \sin 0 = 2\pi^2 + C,$$

which implies that $C = -2\pi^2$, and so the special solution is

$$y^3 + \sin y = 2t^2 - 2\pi^2,$$

Orthogonal trajectories: Consider a family of curves $f(x, y) = 0$. To find the Orthogonal trajectories,

1) Calculate y' from f(x,y)=0

2) Solve the equation $\frac{dy}{dx} = -\frac{1}{y'}$.

Example. Find the Orthogonal trajectories of a family of curves $y = c/x$.

Mixing Problem.

Example. A water reservoir is fed partly by clean water with the speed 0.9 ML/day, and partly by salt water with the speed 0.1 ML/day, where the concentration of salt water is 50 kg/ML. At the bottom of reservoir there is a hole to let the water go out with the speed 1 ML/day. The volume of the reservoir is 100 ML. Assume that the water in the reservoir is clean at beginning. Let $Q(t)$ be the quantity of salt in the reservoir at time t (day).

(1) Construct the differential equation for $Q(t)$.

(2) Solve the differential equation.

(3) Predict the long term quantity of the salt in the reservoir.

Solution: (1)

Rate of change of total quantity of salt

= Rate In - Rate Out

= salt IN per day -salt OUT per day

= salt water volume IN per day × concentration of salt water IN

- salt water volume OUT per day × concentration of salt water OUT.

We have

$$\frac{dQ}{dt} = 0.1 \times 50 - 1 \times \frac{Q}{100}, \Rightarrow \frac{dQ}{dt} = \frac{500 - Q}{100}, \quad Q(0) = 0.$$

(2)

$$Q(t) = 500(1 - e^{-0.01t}).$$

(3)

$$\lim_{t \to \infty} Q(t) = 500.$$

7.4 Exponential Growth and Decay

If the population growth $\frac{dP}{dt}$ is proportional to the population size $P(t)$, then

$$\frac{dP}{dt} = kP.$$

The solution is

$$P(t) = P(0)e^{kt}.$$

$k > 0 \rightarrow$ growth, $k < 0 \rightarrow$ decay.

- Absolute growth rate = rate of change of population = dP/dt,
- Relative growth rate = percent change per unit time = $(dP/dt)/P$,
- Doubling time (D) of exponential growth = the time required for it to double:

$$P(t) = P(0)2^{t/D}.$$

- Half life (H) of exponential growth = the time required for it to be half:

$$P(t) = P(0)\left(\frac{1}{2}\right)^{t/H}.$$

- Newtons Law of Cooling states that the rate of change of the temperature of a cooling body is proportional to the difference between its temperature T and the temperature of its surrounding medium T_s. The model is:

$$\frac{dT}{dt} = -k(T - T_s),$$

where k is a constant.

Example. A bacteria culture growth at a rate proportional to its size. After 2 hours there are 40 bacteria and after 4 hours the count is 120. Find an expression for the population after t hours.

Solution: We measure the time t in hours. Let $P(t)$ be the population at t hours, then we have

$$\frac{dP}{dt} = kP.$$

The solution of the equation is

$$P(t) = P(0)e^{kt}.$$

Note that $P(2) = 40$ and $P(4) = 120$, we obtain

$$40 = P(0)e^{2k}, \quad 120 = P(0)e^{4k}.$$

These imply that

$$P(0) = \frac{40}{3}$$

and

$$e^{2k} = 3, \quad \text{or} \quad k = \ln 3/2.$$

We thus have

$$P(t) = \frac{40}{3}3^{t/2} = \frac{40}{3}\sqrt{3}^t = \frac{40}{3}e^{(\ln 3/2)t}.$$

Example. The half-life of Sodium-24 is 15 hours. Suppose you have 100 grams of Sodium-24. How many grams remaining after 27 minutes (keep three decimals)?

Solution: Assume $m(t)$ be the amount after t hours. Then

$$m(t) = m(0)(\frac{1}{2})^{t/H},$$

where $m(0) = 100$, $H = 15$ hours. Note that 27 minutes = 27/60=0.45 hours. Thus

$$m(0.45) = 100(\frac{1}{2})^{0.45/15} = 100(\frac{1}{2})^{0.03} = 97.942g.$$

Example. Use Newton's Law of Cooling to determine the time of death of a healthy man.

- He died in his room some time before noon;
- At noon, his body temperature was found to be 70 degrees;
- His body cooled another 5 degrees in 1 hour after noon;
- The room temperature was a constant 60 degrees;
- Normal temperature of people's body is 98.6 degree.

Solution: Let $T(t)$ be the temperature of the body at time t. Taking noon as $t = 0$, we have $T(0) = 70$. Note that $T_s = 60$, we have

$$\frac{dT}{dt} = -k(T - 60),$$

this equation is separable. We obtain

$$\frac{dT}{T - 60} = -kdt, \Rightarrow T(t) = 60 + Ce^{-kt}.$$

From $T(0) = 70$ we get $70 = 60 + C$, $C = 10$ and

$$T(t) = 60 + 10e^{-kt}.$$

At 1:00pm, his body temperature is $70 - 5 = 65$. Hence $T(1) = 65$.

$$65 = 60 + 10e^{-k(1)}, \Rightarrow 5 = 10e^{-k}, \Rightarrow k = \ln 2, \Rightarrow T(t) = 60 + 10e^{-t \ln 2}.$$

Thus

$$98.6 = 60 + 10e^{-t \ln 2}, \Rightarrow t = -\ln 3.86 / \ln 2 = -1.95,$$

which means 1 hour and 0.95(60) minutes before noon, or 1 hour and 57 minutes before noon. Or the time of death is 10:03AM.

Interest

Let A_0 = initial amount, r = annual interest rate, $A(t)$ = the amount after t (e.g. years). If the interest is compounded

1. n times per year, then
$$A(t) = A_0(1 + \frac{r}{n})^{nt},$$
2. continuously, then
$$A(t) = A_0 e^{rt}.$$

Example. Invest \$ 1000 for 3 years at 6% interest. Find the value of investment if the interest is compounded

1. daily,
2. continuously.

Solution: 1)

$$A(3) = 1000(1 + \frac{0.06}{365})^{(365)3} = 1197.20.$$

2)

$$A(3) = 1000e^{(0.06)3} = 1197.22.$$

Equilibrium (Additional contents)

Consider the differential equation

$$\frac{dy}{dt} = f(y).$$

Definition: The equilibria or constant solutions of this differential equation are the roots of the equation $f(y) = 0$.

These constant solutions will cut the entire plane (where the solutions live) into independent regions. This means, if an initial condition belongs to one of the regions, then the solution satisfying the initial condition will stay in that region all the time.

Definition. An equilibrium is considered stable if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable. An equilibrium may be stable or unstable. For example, the equilibrium of a pencil standing on its tip is unstable; the equilibrium of a picture on the wall is (usually) stable.

Stability Theorem. Consider the differential equation

$$\frac{dy}{dt} = f(y)$$

with an equilibrium $y*$. If $df(y*)/dy < 0$, then $y*$ is stable; if $df(y*)/dy > 0$, then $y*$ is unstable.

Example. Consider the differential equation

$$\frac{dy}{dt} = y(1 - y).$$

Let $f(y) = y(1 - y)$. From $f(y) = 0$ we have $y = 0$ and $y = 1$. Note that $f'(y) = 1 - 2y$, $f'(0) = 1 > 0$, $f'(1) = -1 < 0$. Hence $y = 0$ is unstable, and $y = 1$ is stable.

7.5 The Logistic Equation

Logistic Differential Equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where k is the relative growth rate, M is the carrying capacity. By Partial Fraction,

$$\int \left(\frac{1}{P} - \frac{\frac{1}{M}}{1 - \frac{P}{M}}\right) dP = \int k dt.$$

Its solution is:

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \quad A = \frac{M - P_0}{P_0}.$$

Properties:

1.
$$P'' = k^2 P \left(1 - \frac{P}{M}\right)\left(1 - \frac{2}{M}P\right);$$

2. $y = M$ is the horizontal asymptote;

3. If $P(0) = M$, then $P(t) = M$ for any t;

4. If $P(0) > M$, then $P(t) > M$, $P(t)$ is decreasing and concave up;

5. If $\frac{M}{2} < P(0) < M$, then $P(t) < M$ for any t, $P(t)$ is increasing and concave down;

6. If $0 < P(0) < \frac{M}{2}$, then $P(t) < M$ for any t, $P(t)$ is increasing and concave up when $P(t) < M/2$, then concave down when $P(t) > M/2$. The t such that $P(t) = M/2$ will be the point of inflection;

7.
$$\lim_{t\to\infty} P(t) = M.$$

Example. Let the population $P(t)$ satisfy

$$\frac{dP}{dt} = 0.2P - 0.000625P^2, \quad P(0) = 80.$$

Which of the following statement(s) will be true?

(1) Find the solution.

(2) Has $P(t)$ any point of inflection?

(3) Predict the long term population.

Sol: (1) We rewrite the DE as:

$$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{320}\right), \quad P(0) = 180.$$

Hence $k = 0.2$, $M = 320$, $A = \frac{320-80}{80} = 3$. Thus

$$P(t) = \frac{320}{1 + 3e^{-0.2t}}$$

(2) Since $P(0) = 80 < M/2 = 160$, $P(t)$ has a point of inflection.

(3) $\lim_{t\to\infty} P(t) = 320$.

Example. Assume that the US population satisfying logistic growth. We know that the population was 250 million in 1990 and 275 million in 2000. Assume the capacity is 2750 million.

1. Construct the differential equation (keep three decimals).
2. Predict the population in 2010.
3. Sketch the graph.

Solution: (a) Let $P(t)$ be the population at t years, $t = 0 \leftrightarrow 1990$. Then we have

$$P(0) = 250, \quad P(10) = 275, \quad P(20) =?$$

The differential equation is

$$\frac{dP}{dt} = kP(1 - \frac{P}{M}).$$

Note that

$$A = \frac{M - P(0)}{P(0)} = \frac{2750 - 250}{250} = 10,$$

the solution is

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{2750}{1 + 10e^{-kt}}.$$

Hence

$$P(10) = 275 = \frac{2750}{1 + 10e^{-10k}}.$$

We have $k = -\ln 0.9/10 = 0.011$. Therefore, the differential equation is:

$$\frac{dP}{dt} = 0.011P(1 - \frac{P}{2750}).$$

(b) The solution is

$$P(t) = \frac{2750}{1 + 10e^{-0.011t}}, \Rightarrow P(20) = \frac{2750}{1 + 10e^{-0.011(20)}} = 304.675$$

Chapter 8 – Infinite Sequences and Series

8.1 Sequences

Sequence:

$$a_1, a_2, ..., a_n, ...$$

a_n is the nth term. If $\lim_{n\to\infty} a_n$ exists, then we say the sequence converges. Otherwise, we say the sequence diverges.

Example. The sequence $\left\{\frac{\ln n}{n}\right\}$ is convergent.

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Example.

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r| < 1; \\ \infty, & |r| > 1. \end{cases}.$$

Thus the sequence $\left\{(\frac{1}{3})^n\right\}$ is convergent, but the sequence $\{(3)^n\}$ is divergent.

Property: If $a_n = f(n)$ and $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Properties: Let $\{a_n\}$ and $\{b_n\}$ be convergent, $c, d \in \mathbb{R}$.

1. $\lim\limits_{n\to\infty} (ca_n + db_n) = c \lim\limits_{n\to\infty} a_n + d \lim\limits_{n\to\infty} b_n$.
2. $\lim\limits_{n\to\infty} (a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$.
3. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $\lim\limits_{n\to\infty} b_n \neq 0$.
4. $\lim\limits_{n\to\infty} a_n^p = (\lim\limits_{n\to\infty} a_n)^p$, if $a_n \geq 0$ and $p > 0$.
5. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Squeeze Theorem: If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Example. $\{\frac{\cos n}{n}\}$ converges 0.

Bounded above: $a_n \le M$ for all n.
Bounded below: $a_n \ge m$ for all n.
Bound and Convergence: A convergent sequence is bounded.
Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.
Example. $a_n = \frac{n}{n^3+1}$ is decreasing, and $0 < a_n < 1$ for any n.

To check it, let $f(x) = \frac{x}{x^3+1}$. Then $f'(x) = \frac{1-2x^2}{(x^3+1)^2} < 0$ when $x \ge 1$. Hence $f(x)$ is decreasing when $x \ge 1$.

8.2 Series

A basic fact is

$$\lim_{n\to\infty} r^n = 0 \Leftrightarrow |r| < 1.$$

The sum $\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + ...$ is called infinite series.
Partial sum:

$$S_n = \sum_{j=1}^{n} a_j.$$

Then

$$\sum_{j=1}^{\infty} a_j = S \Leftrightarrow \lim_{n\to\infty} S_n = S.$$

- Geometric series: if $|r| < 1$ then $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$.
- $\frac{k}{n(n+k)} = \frac{1}{n} - \frac{1}{n+k}$.
- Harmonic series $\sum_{j=1}^{\infty} \frac{1}{n}$ is divergent.
- Divergence Test: If $\lim_{n\to\infty} a_n \ne 0$, then $\sum_{j=1}^{\infty} a_j$ is divergent.
- $\sum_{j=1}^{\infty}(ca_j + db_j) = c\sum_{j=1}^{\infty} a_j + d\sum_{j=1}^{\infty} b_j$.

Example. Determine if the series converges or diverges:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+2}}{3^{n+1}}$

(b) $\sum_{n=1}^{\infty} 3^{n+1} 2^{-2n}$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{\ln n}$.

(d) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$.

Solution:(a)

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{2^{2(n+1)}}{3^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^{n+1}.$$

It is a geometric series with ratio $r = 4/3 > 1$. Therefore it is divergent.

(b) We have

$$\sum_{n=1}^{\infty} 3^{n+1} 2^{-2n} = \sum_{n=1}^{\infty} 3 \cdot 3^n 4^{-n} = 3 \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

This series is a geometric series with first term $a = 9/4$ and common ratio $r = 3/4$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 3^{n+1} 2^{-2n} = \frac{a}{1-r} = \frac{9/4}{1/4} = 9.$$

(c) Divergent by Divergence Test.

(d)

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots\right) = \frac{5}{12}.$$

8.3 The Integral and Comparison Test; Estimating sums

Integral test: If $f(x)$ is continuous, positive, decreasing, $f(j) = a_j$, then

$$\sum_{j=1}^{\infty} a_j \text{ is convergent} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ is convergent.}$$

Example. Determine if the series converges or diverges $\displaystyle\sum_{n=3}^{\infty} \frac{3}{n(\ln n)^3}$

Sol: We use Integral Test. Let $f(x) = \frac{3}{x(\ln x)^3}$. Then $f(x) > 0$ for $x \geq 3$. Since

$$f'(x) = \frac{-3[(\ln x)^3 + 3(\ln x)^2]}{x^2(\ln x)^6} < 0,$$

$f(x)$ is decreasing. By substitution with $u = \ln x$, we have

$$\begin{aligned}\int_3^\infty \frac{3}{x(\ln x)^3}dx &= \lim_{b\to\infty}\int_3^b \frac{3}{x(\ln x)^3}dx \\ &= \lim_{b\to\infty}\int_{\ln 3}^{\ln b}\frac{3}{u^3}du \\ &= \frac{3}{2}(\frac{1}{\ln 3})^2.\end{aligned}$$

Hence it is convergent.

The $p-$series: $\sum_{j=1}^\infty \frac{1}{j^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example. For what value of p is the series

$$\sum_{n=1}^\infty n^{p-1}$$

convergent?

Solution:

$$\sum_{n=1}^\infty n^{p-1} = \sum_{n=1}^\infty \left(\frac{1}{n}\right)^{-p+1}.$$

Hence, if $-p + 1 > 1$, i.e., $p < 0$, then the series is convergent.

Comparison test: if $0 \leq a_j \leq b_j$, then

$$\sum_{j=1}^\infty b_j \text{ is convergent} \Rightarrow \sum_{j=1}^\infty a_j \text{ is convergent.}$$

$$\sum_{j=1}^\infty a_j \text{ is divergent} \Rightarrow \sum_{j=1}^\infty b_j \text{ is divergent.}$$

Example. Determine if the series converges or diverges $\sum_{n=1}^\infty \frac{1}{n(n^3+5)^{1/3}}$.

Solution: By Comparison test,

$0 \leq \sum_{n=1}^\infty \frac{1}{n(n^3+5)^{1/3}} \leq \sum_{n=1}^\infty \frac{1}{n^2} < \infty$ (p-series, $p = 2 > 1$), so convergent.

The Limit Comparison test: Suppose that $a_j > 0$ and $b_j > 0$ for any j, and

$$\lim_{j\to\infty} \frac{a_j}{b_j} = c > 0.$$

Then either both $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are convergent or both diverge.

Example. Determine if the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{3n^3+2n+1}{9n^3\sqrt{n}+1}.$$

Solution. Note that

$$a_n = \frac{3n^3+2n+1}{9n^3\sqrt{n}+1} \sim \frac{3n^3}{9n^3\sqrt{n}} = \frac{1}{3\sqrt{n}}.$$

Let

$$b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}.$$

Then

$$\sum_{n=1}^{\infty} b_n$$

is divergent by p-series test with $p = 1/2$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3},$$

the given series is divergent by Limit Comparison Test.

Remainder estimate: Let $R_n = \sum_{j=n+1}^{\infty} a_j$. Then

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx.$$

Example. (1) Estimate the error if we use the sum of the first 10 terms to approximate

$$\sum_{n=1}^{\infty} \frac{3}{n^4}.$$

(2) Find the smallest n such that S_n is within 0.000001 of the sum.

Solution. (1) Let $f(x) = \frac{3}{x^4}$. Then

$$\int_{11}^{\infty} f(x)dx \le R_{10} \le \int_{10}^{\infty} f(x)dx, \Rightarrow \frac{1}{11^3} \le R_{10} \le \frac{1}{10^3}.$$

(2)

$$R_n \le \frac{1}{n^3}, \Rightarrow \frac{1}{n^3} \le 0.000001, \Rightarrow n \ge 100,$$
$$R_{99} \ge \int_{100}^{\infty} f(x)dx = 0.000001.$$

8.4 Other Convergence Tests

1. Alternating series Test: The alternating series $\sum_{j=1}^{\infty}(-1)^{j-1}b_j$ is convergent if

1. $b_j > 0$
2. b_j decreasing ($b_1 \ge b_2 \ge b_3 \ge \cdots$)
3. $\lim_{j\to\infty} b_j = 0$.

For the alternating series, we can estimate the **remainder** by using the following inequality

$$|R_n| \le b_{n+1}.$$

Example. Test the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n^2+n+1}$ for convergence or divergence.

Solution: This is an alternating series. Let $b_n = \frac{n+1}{n^2+n+2}$. Then

1) $\lim_{n\to\infty} b_n = 0$;

2) Let $f(x) = \frac{x+1}{x^2+x+1}$ for $x \ge 1$, then

$$f'(x) = \frac{-x^2 - 2x}{(x^2+x+1)^2} < 0.$$

Therefore $f(x)$ decreases for $x \ge 1$. In particular,

$$f(n) > f(n+1)$$

for all positive integer n. Hence b_n decreases.

By the Alternating Series Test, the series converges.

2. Absolute convergence: If the series $\sum_{j=1}^{\infty}|a_j|$ is convergent, then we say that $\sum_{j=1}^{\infty} a_j$ is absolutely convergent; If the series $\sum_{j=1}^{\infty} a_j$ is convergent, but $\sum_{j=1}^{\infty}|a_j|$ is divergent, then we say that $\sum_{j=1}^{\infty} a_j$ is conditionally convergent.

Property:

$$\sum_{j=1}^{\infty} |a_j| \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

Example. Determine whether the series is absolutely convergent: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n^2+n+1}$.

Solution:

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n+2}{n^2+n+1}\right| = \sum_{n=1}^{\infty}\frac{n+2}{n^2+n+1}.$$

Note that

$$\frac{n+2}{n^2+n+1} \geq \frac{n}{3n^2} = \frac{1}{3n}.$$

The series

$$\sum_{n=1}^{\infty}\frac{1}{3n}$$

is divergent (the harmonic series), by Comparison Theorem, the original series is not absolutely convergent.

By Alternating Series Test, the series is convergent, therefore the series is conditionally convergent.

3. Ratio test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty}|\frac{a_{j+1}}{a_j}| = L$.

- If $L < 1$, then the series is absolutely convergent;
- If $L > 1$, then the series is divergent.

Example. Test the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ for convergence or divergence.

Solution: Let $a_n = \frac{n!}{n^n}$. Then

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1.$$

By the Ratio Test, the series converges.

8.5 Power Series

Definition. The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called power series, a is called the center. There exists $R \geq 0$ such that the series is convergent in $|x-a| < R$ and divergent in $|x-a| > R$. R is called radius of convergence.

- Radius of convergence: Using Ratio Test to find R. If
$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = k|x-a|,$$
then $k|x-a| < 1$, so $|x-a| < 1/k = R$.

- Interval of convergence I: Symmetric to the center a, with two end points $a-R$ and $a+R$. The convergence or divergence at the two end points $x = a-R$ and $x = a+R$ should be checked.

Example. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n}$.

Sol: We use Ratio Test.
$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)\,5^{n+1}} / \frac{(-1)^n(x-1)^n}{n\,5^n}| = \lim_{n\to\infty} \frac{1}{5}\frac{n}{n+1}|x-1|$$
$$= \frac{1}{5}|x-1| = L.$$
When $L < 1$, we have $|x-1| < 5$. Hence
$$R = 5,$$
and $-4 < x < 6$.

When $x = -4$,
$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-4-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$
which is divergent (Harmonic series).

When $x = 6$,
$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(6-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$
which is convergent by Alternating Series Test. Therefore,
$$I = (-4, 6].$$

Example. Find the radius and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2}$$

Solution: Let

$$a_n = \frac{(2x-3)^n}{n^2}.$$

Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot |(2x-3)| = |2x-3|.$$

By Ratio Test, $|2x-3| < 1$, i.e., $|x-1.5| < 0.5$. Therefore $R = 0.5$ and $1 < x < 2$.

When $x = 1$,

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which is convergent by Alternating Series Test.

When $x = 2$,

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is convergent by $p-$Series Test. Therefore,

$$I = [1, 2].$$

Example. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{n\, 8^n}$.

Sol: We use Ratio Test.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{1}{8} \frac{n}{n+1} |x-1|^3$$

$$= \frac{1}{8} |x-1|^3.$$

By Ratio Test, we have $|x-1|^3 < 8$, i.e., $|x-1| < 2$. Hence

$$R = 2, \quad and \quad -1 < x < 3.$$

When $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{n\, 8^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent (Harmonic series).

When $x = 3$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{n\, 8^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

which is convergent by Alternating Series Test. Therefore,

$$I = (-1, 3].$$

8.6 Representations of Functions as Power Series

- Basic result:
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$
- Term-by-term differentiation: If
$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$
then
$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$
and f and f' have the same radius of convergence.
- Term-by-term integration: If
$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$
then
$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1},$$
and f and $\int f dx$ have the same radius of convergence.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $\frac{1}{1+x}$.

(ii)

$$\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}, \quad |x| < 1.$$

(iii)

$$\frac{x^3}{6x+3} = \frac{x^3}{3}\frac{1}{1+2x} = \frac{x^3}{3}\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{3}(x)^{n+3}.$$

Method 1: This series is convergent when $|-2x| < 1$, which is $|x| < 0.5$. So $I = (-0.5, 0.5)$.

Method 2: Let $a_n = (-1)^n \frac{2^n}{3}(x)^{n+3}$. Then

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{2^{n+1}}{3}(x)^{n+4}}{(-1)^n\frac{2^n}{3}(x)^{n+3}}\right| = 2|x|.$$

By the Ratio Test, when $L < 1$, we have $2|x| < 1$, $|x| < 0.5$. Hence $R = 0.5$. When $x = \pm 0.5$, $(-1)^n \frac{2^n}{3}(x)^{n+3} \not\to 0$, the series diverge. So $I = (-0.5, 0.5)$.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $f(x) = \frac{1}{(1+x)^2}$.

$$\frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) = \frac{d}{dx}[(-1)\sum_{n=0}^{\infty}(-x)^n] = \sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}, \quad -1 < x < 1.$$

(ii) $f(x) = \frac{1}{(1-x)^3}$.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $f(x) = \ln(1+x)$.

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$, $R = 1$

(ii) $f(x) = \arctan x$.

$$\arctan x = \int \frac{1}{1+x^2}dx = \int (\sum_{n=0}^{\infty}(-1)^n x^{2n})dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad -1 < x < 1.$$

Note that $C = \arctan 0 = 0$, we have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad -1 < x < 1.$$

Example. (i) Represent $\int \frac{1}{1+x^3}dx$ as power series and use it to approximate $\int_0^{0.1} \frac{1}{1+x^3}dx$ correct to within 0.00001.

8.7 Taylor and Maclaurin Series

- Taylor series:
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n,\quad |x-a|<R.$$
- Maclaurin series:
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n,\quad |x|<R.$$
- Taylor polynomial:
$$T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i.$$
$$f(x)\approx T_n(x).$$
- Taylor's inequality: If $|f^{(n+1)}(x)|\le M$ for $|x-a|\le d$, then on $|x-a|\le d$,
$$|R_n(x)|=|f(x)-T_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}.$$
- Taylor Theorem: If $f(x)=T_n(x)+R_n(x)$, and $\lim_{n\to\infty}R_n(x)=0$ for $|x-a|<R$, then $f(x)$ is equal to its Taylor series for $|x-a|<R$. To this end, the following result is useful:
$$\lim_{n\to\infty}\frac{x^n}{n!}=0.$$
- Multiplication and division of power series

Example. Maclaurin series for some special functions:

$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,\ R=\infty;$

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots,\ R=\infty;$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots, R=\infty;$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,\ R=1;$

$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots,\quad -1<x<1.$

Example.
$$e^{\sin x}=1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\ldots$$

$$= 1 + (x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...) + (\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^2}{2!}) + (\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^3}{3!}) + ...$$

$$= 1 + x + \frac{x^2}{2!} + 0x^3 + ...$$

Binomial series If k is a real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} x^n,$$

here

$$\begin{pmatrix} k \\ n \end{pmatrix} = \frac{k(k-1)\cdots(k-n+1)}{n!}, \quad \begin{pmatrix} k \\ 0 \end{pmatrix} = 1.$$

Application: Let $f(x) = (1+x)^k$, then

$$f^{(n)}(0) = \begin{pmatrix} k \\ n \end{pmatrix} n! = k(k-1)\cdots(k-n+1).$$

Example. Let $f(x) = \sqrt[5]{1+x^2}$. Evaluate $f^{(4)}(0)$.

Solution. Use the binomial series to find the Maclaurin series of $f(x)$.

$$\frac{f^{(n)}(0)}{n!} = \begin{pmatrix} k \\ n \end{pmatrix}$$

Hence

$$f^{(n)}(0) = \begin{pmatrix} k \\ n \end{pmatrix} n! = k(k-1)\cdots(k-n+1).$$

So $f^{(4)}(0) = -0.8064$.

Example. Evaluate

$$\lim_{x \to 0} \frac{e^x \sin x - x}{x^2}.$$

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(x - \frac{x^3}{3!} + ...\right)$$

$$= x + x^2 + \frac{1}{3}x^3 + ...$$

Therefore

$$\frac{e^x \sin x - x}{x^2} = 1 + \frac{1}{3}x + ...$$

Example. Let

$$f(x) = \int_0^x t^3 e^{3t} dt$$

(a) Find the Maclaurin series of the function f.

(b) Find the radius of the series in (a).
Solution: a)

$$t^3e^{3t} = t^3\sum_{n=0}^{\infty}\frac{(3t)^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n}{n!}t^{n+3}.$$

Hence

$$f(x) = \int_0^x\sum_{n=0}^{\infty}\frac{3^n}{n!}t^{n+3}dt$$

$$= \sum_{n=0}^{\infty}\int_0^x\frac{3^n}{n!}t^{n+3}dt = \sum_{n=0}^{\infty}\frac{3^n}{n!}\frac{t^{n+4}}{n+4}|_0^x$$

$$= \sum_{n=0}^{\infty}\frac{3^n}{n!(n+4)}x^{n+4}$$

b)

$$|a_{n+1}/a_n| = \frac{3(n+4)}{(n+1)(n+5)}|x| \to 0$$

as $n \to \infty$. Therefore $R = \infty$.

Example. Find the first 5 non-zero terms in the Maclaurin series for $e^x\cos(3x)$.

Solution. Note that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Hence

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + ...$$

$$= 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - ...$$

We have

$$e^x\cos(3x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$$

$$-\frac{9x^2}{2} - \frac{9x^3}{2} - \frac{9x^4}{4} - ...$$

$$+\frac{27x^4}{8} + ...$$

$$= 1 + x + (\frac{1}{2} - \frac{9}{2})x^2 + (\frac{1}{6} - \frac{9}{2})x^3 + (\frac{1}{24} - \frac{9}{4} + \frac{27}{8})x^4 + ...$$
$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \frac{7}{6}x^4 + ...$$

Applications of Taylor series.

Example.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

Example. Calculate the sum of the following series, given that it is a known series evaluated at a certain value of x:

$$1 - \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} - \frac{1}{64 \cdot 6!} + \frac{1}{256 \cdot 8!} - \cdots$$

Sol:

$$1 - \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} - \frac{1}{64 \cdot 6!} + \frac{1}{256 \cdot 8!} - \cdots$$
$$= 1 - \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 - \frac{1}{6!}\left(\frac{1}{2}\right)^6 + \frac{1}{8!}\left(\frac{1}{2}\right)^8 - \cdots$$
$$= \cos\frac{1}{2}.$$

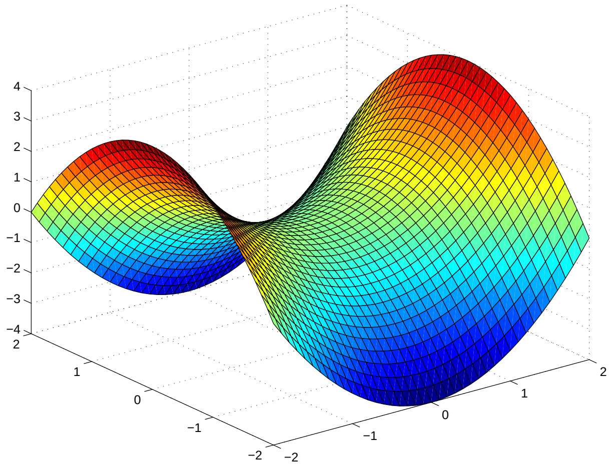
Chapter 9 – Vectors and the Geometry of Space

9.6 Functions and Surfaces

Definition: A function of two variables $z = f(x, y)$ is a relation which maps each point (x,y) in a set D in the xy-plane to a unique number z. The set D is called the domain of the function, which is often denoted D(f).

$\{(x, y, z) : (x, y) \in D, z = f(x, y)\}$ is called the graph of f (which is a surface).

Example. Sketch the graph of the function $z = x^2 - y^2$ (Hyperbolic Paraboloid):



Example. Classify the following quadric surfaces: $x^2 + 2z^2 - 6x - y + 10 = 0$.

Solution: We change it to $\frac{(x-3)^2}{\sqrt{2}^2} + \frac{z^2}{1^2} = \frac{y-1}{2}$. This is an elliptic paraboloid with vertex (3, 1, 0), centered with the line x = 3, y = 1.

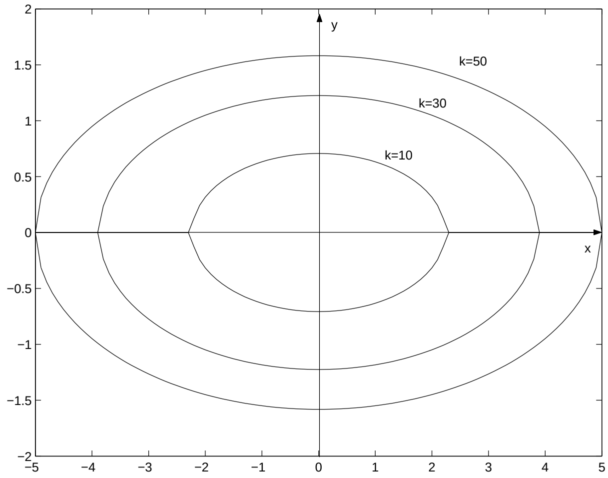
Example. Find the domain of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Solution: $4 - x^2 - y^2 \geq 0$, i.e., $x^2 + y^2 \leq 4$. Thus $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

Chapter 11 – Partial Derivatives

11.1 Functions of Several Variables

Functions of two variables: $z = f(x,y)$.
Level curves (contour maps) of $f(x,y)$: $f(x,y) = k$ for different k.
Example. Let $f(x,y) = 2x^2 + 20y^2$. Sketch three level curves for $k = 10, 30, 50$.



Functions of three variables: A function of three variables $w = f(x,y,z)$ is a rule which maps each point (x,y,z) in a set D to a unique number w. The set D is called the domain of the function, which is often denoted D(f).
Example. Find the domain of the function $f(x,y,z) = \sqrt{4 - x^2 - y^2 - z^2}$.

Solution: $4 - x^2 - y^2 - z^2 \geq 0$, i.e., $x^2 + y^2 + z^2 \leq 4$. Thus $D = \{(x,y) : x^2 + y^2 + z^2 \leq 4\}$.
$w = f(x,y,z)$.
Level surfaces of $f(x,y,z)$: $f(x,y,z) = k$ for different k.

11.3 Partial Derivatives

Functions of two variables

- Partial derivatives of $z = f(x, y)$:

$$z_x = \frac{\partial z}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y) := D_x f := \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

  which is the derivative of f with respect to x;

$$z_y = \frac{\partial z}{\partial y} := \frac{\partial f}{\partial y} := f_y(x, y) := D_y f := \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

  which is the derivative of f with respect to y.

- Methods:
  1. To find f_x: regard y as a constant, and differentiate $f(x, y)$ with respect to x;
  2. To find f_y: regard x as a constant, and differentiate $f(x, y)$ with respect to y.

- Meaning: f_x means the rate of change of f with respect to x when y is fixed.

Example. Let $f(x, y) = e^{xy} + \frac{x}{y}$. Calculate $f_x(0, 1)$, $f_y(0, 1)$.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, \quad f_x(0, 1) = 2.$$

$$f_y = xe^{xy} - \frac{x}{y^2}, \quad f_y(0, 1) = 0.$$

Functions of three variables

- Let $w = f(x, y, z)$, then

$$\frac{\partial w}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y, z) := D_x f := \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h},$$

  which is the derivative of f with respect to x.

- Meaning: f_x means the rate of change of f (or w) with respect to x when y and z are fixed.

Example. Let $f(x,y,z) = \sin z e^{xy} \ln x$. Calculate f_x, f_y, f_z.

Implicit differentiation:

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if z is implicitly defined by

$$x^2 + y^3 + z^4 - 8xyz = 1.$$

Higher derivatives:

f_{xx}, $\frac{\partial^3 f}{\partial z \partial y \partial x} = f_{xyz}, \cdots$

Example. Let $f(x,y) = e^{xy} + \frac{x}{y}$. Calculate f_{xx}, f_{xy}, f_{yy}.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, f_y = xe^{xy} - \frac{x}{y^2}.$$

$$f_{xx} = y^2 e^{xy}, \quad f_{xy} = e^{xy} + xye^{xy} - \frac{1}{y^2}, \quad f_{yy} = x^2 e^{xy} + \frac{2x}{y^3}.$$

Example. Let $f(x,y,z) = \sin z e^{xy} \ln x$. Calculate f_{xyz}.

Partial differential equations:

1. Laplace equation:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
2. Wave equation:
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Example. Verify that $u(x,y) = \sin(x - 3t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}.$$

11.4 Tangent Planes and Linear Approximations

1. Functions of two variables.

- Equation of the tangent plane of $z = f(x, y)$ at (x_0, y_0):

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Linear Approximation (Tangent plane approximation):

$$\begin{aligned} f(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= L(x, y). \end{aligned}$$

- Differential notation: $dz = f_x(x, y)dx + f_y(x, y)dy$. If we take $dx = x - x_0$, $dy = y - y_0$, then $dz|_{(x_0,y_0)} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Example. Find the equation of the tangent plane of the surface $z = e^{xy} + \frac{x}{y}$ at the point $(2, 1, e^2 + 2)$.

Solution. Let $f(x, y) = e^{xy} + \frac{x}{y}$. Then

$$f_x = ye^{xy} + \frac{1}{y}, \quad f_x(2, 1) = e^2 + 1.$$

$$f_y = xe^{xy} - \frac{x}{y^2}, \quad f_y(2, 1) = 2e^2 - 2.$$

Thus the equation of the tangent plane at the point $(2, 1, e^2 + 2)$ is

$$z - (e^2 + 2) = (e^2 + 1)(x - 2) + (2e^2 - 2)(y - 1), \quad i.e., \quad z = (e^2 + 1)x + (2e^2 - 2)y - 3e^2 + 2.$$

Example. Use the linear approximation of $f(x, y) = 2x^2y^2 + 3xy + x$ at $(1, 1)$ to approximate $f(0.9, 1.1)$.

Solution.

$f(x, y) \approx f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$.

$f(x, y) = 2x^2y^2 + 3xy + x,\ f_x = 4xy^2 + 3y + 1,\ f_y = 4x^2y + 3y$

$f(1, 1) = 6,\ f_x(1, 1) = 8,\ f_y(1, 1) = 7$

Thus $f(x, y) \approx 6 + 8(x - 1) + 7(y - 1)$.

$f(0.9,\ 1.1) \approx 6 + 8 * (-0.1) + 7 * 0.1 = 5.9$

Example. Let $z = x^2 + 3xy - y^2$.

(a) Find dz.

(b) If x changes from 2 to 2.05, y changes from 3 to 2.96, calculate dz and Δz.

Solution: (a) $dz = f_x(x, y)dx + f_y(x, y)dy = (2x + 3y)dx + (3x - 2y)dy$.

(b) Putting $x_0 = 2$, $dx = 2 - 2.05 - 2 = 0.05$, $y_0 = 3$, $dy = 2.96 - 3 = -0.04$, then

$$dz|_{(x_0,y_0)} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.65.$$

$$\Delta z = f(2.05, 2.96) - f(2, 3) = 0.6449.$$

2. Functions of three or more variables $w = f(x, y, z)$**.**

- Linear Approximation (Tangent plane approximation) at (x_0, y_0, z_0):

$$\begin{aligned} f(x, y, z) &\approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ &\quad + f_z(x_0, y_0, z_0)(z - z_0) \\ &= L(x, y, z). \end{aligned}$$

- Equation of the tangent plane of to the level surface $f(x, y, z) = k$ at (x_0, y_0, z_0) is: $f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$.
- Differential: $dw = w_x dx + w_y dy + w_z dz$.

Example. Find the equation of the tangent plane to the level surface $2x^2+y^3+z^4-xyz = 11$ at $(1, 1, -1)$.

11.5 The Chain Rule

- Basic Chain Rule

  1. If $z = f(x, y)$, $x = g(t)$, $y = h(t)$, then
  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
  2. $\frac{dz}{dt}$ means rate of change of z with respect to t along the path $x = g(t)$, $y = h(t)$, $t \in D$.
  3. If $w = f(x, y, z)$, $x = g(t)$, $y = h(t)$, $z = k(t)$, then
  $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

- General Chain Rule: If $w = f(x, y, z)$, $x = g(u, v)$, $y = h(u, v)$, $z = k(u, v)$, then
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

- Implicit Differentiation:

  1. If $F(x, y) = 0$, then
  $$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$
  Here when we calculate partial derivatives, we consider x and y as independent variables.
  2. If $F(x, y, z) = 0$, then
  $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$
  Here when we calculate partial derivatives, we consider x, y and z as independent variables.

Example. Suppose $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$. Given the data

$g(1) = 1,\ g'(1) = 2,$

$h(1) = 2, h'(1) = 3,$

$f_x(1, 2) = -1,\ f_y(1, 2) = 2.$

Find $\frac{dz}{dt}$ when $t = 1$.

Solution.
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x g' + f_y h'$$
$$t = 1 \Rightarrow (x, y) = (g(1), h(1)) = (1, 2)$$
$$\frac{dz}{dt}|_{t=1} = f_x(1, 2)g'(1) + f_y(1, 2)h'(1)$$
$$= (-1)(2) + (2)(3) = 4$$

Example. Consider the following function

$$z = x^2y + e^x \cos y, \quad x = t^3 \sin s,\ y = s^2 + 3t^2.$$

Calculate $\frac{\partial z}{\partial s}$ at the point $(s, t) = (0, 1)$ by using Chain Rule.

Example. Find y' if $x^2y + e^x \cos y = 3$.

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2y^3 + z^4 + 5xyz = 3$.

11.6 Directional Derivatives and the Gradient Vector

Review of vectors in plane

1. A point $P(a, b)$ can be considered as a vector $\overrightarrow{OP}$.
2. Any vector in the plane can be expressed as: $\vec{v} =< v_1, v_2 >= (v_1, v_2) = v_1\vec{i} + v_2\vec{j}$.
3. The length or magnitude of $\vec{v}$ is: $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$. $\vec{v}$ is a unit vector if $|\vec{v}| = 1$.
4. Any non-zero vector $\vec{v}$ can be normalized to a unit vector: $\frac{\vec{v}}{|\vec{v}|} =< \frac{v_1}{|\vec{v}|}, \frac{v_2}{|\vec{v}|} >$.
5. Dot product: $< u_1, u_2 > \cdot < v_1, v_2 >= u_1v_1 + u_2v_2$, perpendicular (orthogonal) if the dot product is zero.

11.6.1 Gradients and Directional Derivatives in the Plane

- Directional derivatives:
  1. The directional derivative of the function $f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\vec{u} =< u_1, u_2 >$ is

$$\begin{aligned} D_{\vec{u}}f(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h} \\ &= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \\ &= < f_x(x_0, y_0), f_y(x_0, y_0) > \cdot < u_1, u_2 > . \end{aligned}$$

  2. $D_{\vec{u}}f(x_0, y_0)$ means the rate of change of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$.

- The gradient of $f(x, y)$ at (x_0, y_0) is

$$\nabla f(x_0, y_0) = gradf(x_0, y_0) =< f_x(x_0, y_0), f_y(x_0, y_0) > .$$

  1. $\nabla f(x_0, y_0)$ points into the direction of maximum increase of f at (x_0, y_0).
  2. $\nabla f(x_0, y_0)$ is perpendicular to the contour line (or level curve) of f through (x_0, y_0).
  3. $|\nabla f(x_0, y_0)|$ is the maximum rate of change of f at (x_0, y_0).

Example. Let $f(x, y) = \sqrt{2x^2 + 3y^2}$.

(1) Sketch three level curves, and calculate the gradient of the function.

(2) Find the directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction of the vector $\langle 1, \sqrt{3}\,\rangle$.

(3) Find the maximum rate of change of f at (2, 1) and indicate in which direction this maximum will occur.

Example. Let $f(x, y) = x^2 y + 4y^2$.

(1) Calculate the gradient of the function.

(2) Find the directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction of the vector $\langle 1, \sqrt{3}\,\rangle$.

(3) Find the maximum rate of change of f at (2, 1) and indicate in which direction this maximum will occur.

Solution. (1) $\nabla f = \langle 2xy, x^2 + 8y\rangle$.

(2) $\vec{v} = \langle 1, \sqrt{3}\,\rangle$, $\vec{u} = \frac{1}{2}\langle 1, \sqrt{3}\,\rangle$,, $\nabla f(2, 1) = \langle 4, 12\rangle$.

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$D_{\mathrm{u}} f(2, 1) = \nabla f(2, 1) \cdot \vec{u} = \langle 4, 12\rangle \cdot \frac{1}{2}\langle 1, \sqrt{3}\,\rangle = 2 + 6\sqrt{3}$

(3) The maximum rate of change of f at (2, 1) $= |\nabla f(2, 1)| = |\langle 4, 12\rangle| = 4\sqrt{10}$, which occurs in the direction $\langle 4, 12\rangle$.

11.6.2 Gradient and Directional Derivatives in space

Review of vector in space

1. A point $P(a, b, c)$ can be considered as a vector $\overrightarrow{OP}$.
2. Any vector in the space can be expressed as: $\vec{v} =< v_1, v_2, v_3 >= (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$.
3. The length or magnitude of $\vec{v}$ is: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. $\vec{v}$ is a unit vector if $|\vec{v}| = 1$.
4. Any non-zero vector $\vec{v}$ can be normalized to a unit vector: $\frac{\vec{v}}{|\vec{v}|} =< \frac{v_1}{|\vec{v}|}, \frac{v_2}{|\vec{v}|}, \frac{v_3}{|\vec{v}|} >$.
5. Dot product: $< u_1, u_2, u_3 > \cdot < v_1, v_2, v_3 >= u_1v_1 + u_2v_2 + u_3v_3$, perpendicular (orthogonal) if the dot product is zero.

- Directional derivatives:

1. The directional derivative of the function $f(x,y,z)$ at (x_0,y_0,z_0) in the direction of a unit vector $\vec{u} =< u_1,u_2,u_3 >$ is

$$f_{\vec{u}}(x_0,y_0,z_0) = D_{\vec{u}}f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)u_1+f_y(x_0,y_0,z_0)u_2+f_z(x_0,y_0,z_0)u_3.$$

- The gradient of $f(x,y,z)$ at (x_0,y_0,z_0) is

$$grad f(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0) =< f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0) > .$$

1. $\nabla f(x_0,y_0,z_0)$ points into the direction of maximum increase of f at (x_0,y_0,z_0).
2. $\nabla f(x_0,y_0,z_0)$ is perpendicular to the level surface of f through (x_0,y_0,z_0).
3. $|\nabla f(x_0,y_0,z_0)|$ is the maximum rate of change of f at (x_0,y_0,z_0).

- Equation of the tangent plane of to the level surface $f(x,y,z)=k$ at (x_0,y_0,z_0) is: $< f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0) > \cdot < x-x_0, y-y_0, z-z_0 >= 0$, i.e., $f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$.

Example. Find the equation of the tangent plane at the point (2,1,9) to the ellipsoid $\frac{x^2}{12}+\frac{y^2}{3}+\frac{z^2}{27}=1$.

Solution: $x+2y+2z-22=0$.