MAT 1341 Solutions to Homework Assignment #1

Question 1. [**1 point**] Consider the plane $P : 3x - 2y + 6z = -3$ and the line $l : (x, y, z) = (3, 4, 0) + t(4, -3, -3)$. Which statement about the intersection of P and l is correct?

A P and l intersect in a unique point, namely $(3, 4, 0)$.
B P and l intersect in a unique point, namely $(4, -3, -3)$.
C P and l intersect in a unique point, namely $(3, -2, 6)$.
D P and l are parallel and have empty intersection.
E P contains l.
F None of the above.

Answer: First note that P and l are parallel, because $(3, -2, 6) \cdot (4, -3, -3) = 0$, meaning that the direction of the line is perpendicular to the normal of the plane. Thus either P and l have empty intersection or P contains l. In the latter case the point of the line should be in P. But $3 \cdot 3 - 2 \cdot 4 + 6 \cdot 6 = 1$, and hence $(3, 4, 0)$ doesn't satisfy the equation of the plane. Conclusion: P and l have empty intersection.

Question 2. [**1 point**] What is the polar form of $z = -\sqrt{3} - i$?

A $z = 2e^{\frac{5\pi i}{4}}$
B $z = \sqrt{2}e^{\frac{5\pi i}{4}}$
C $z = -2e^{\frac{5\pi i}{4}}$
D $z = 2e^{\frac{7\pi i}{6}}$
E $z = \sqrt{2}e^{\frac{7\pi i}{6}}$
F $z = -2e^{\frac{7\pi i}{6}}$

Answer: Draw a picture! Note first that this number lies in the third quadrant. We have $\tan\theta = \frac{-1}{-\sqrt{3}}$, so the argument is $\theta = \frac{7\pi}{6}$. The modulus is $\sqrt{(-1)^2 + \sqrt{3}^2} = 2$. Hence $z = 2e^{\frac{7\pi i}{6}}$. (Note: the modulus is always a positive number.)

Question 3. [**3 points**] Consider the three standard basis vectors $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$. Which of the following are true? Circle ALL that apply.

A $\hat{i} \cdot (\hat{j} \times \hat{k}) = -\hat{i} \cdot (\hat{k} \times \hat{j})$
B $\hat{i} \times (\hat{j} \times \hat{k}) = (\hat{i} \times \hat{j}) \times \hat{k}$
C $\hat{i} \times (\hat{j} \times \hat{i}) = \hat{k} \times (\hat{i} \times \hat{k})$
D $(\hat{i} \times \hat{j}) \times (\hat{k} \times \hat{j}) = -\hat{i} \times \hat{j}$
E $\hat{k} \times (\hat{i} - \hat{k}) = (\hat{k} \times \hat{i}) - (\hat{k} \times \hat{j})$
F $Proj_{\hat{k}}(\hat{i} + \hat{j}) = Proj_{\hat{k}}(\hat{i}) + Proj_{\hat{k}}(\hat{j})$

Answer: Note first that we have

$$\hat{\jmath}\times\hat{k}=\hat{\imath},\quad \hat{\imath}\times\hat{\jmath}=\hat{k},\quad \hat{\imath}\times\hat{k}=-\hat{\jmath}.$$

We also know that $v\times u=-u\times v$. Now
A $\hat{\imath}\cdot(\hat{\jmath}\times\hat{k})=\hat{\imath}\cdot\hat{\imath}=1$ and $-\hat{\imath}\cdot(\hat{k}\times\hat{\jmath})=-\hat{\imath}\cdot-\hat{\imath}=1$.
B $\hat{\imath}\times(\hat{\jmath}\times\hat{k})=\hat{\imath}\times\hat{\imath}=0=\hat{k}\times\hat{k}=(\hat{\imath}\times\hat{\jmath})\times\hat{k}$
C $\hat{\imath}\times(\hat{\jmath}\times\hat{\imath})=\hat{\imath}\times-\hat{k}=\hat{\jmath}$, while $\hat{k}\times(\hat{\imath}\times\hat{k})=\hat{k}\times-\hat{\jmath}=\hat{\imath}$.
D $(\hat{\imath}\times\hat{\jmath})\times(\hat{k}\times\hat{\jmath})=\hat{k}\times-\hat{\imath}=\hat{\jmath}$, while $-\hat{\imath}\times\hat{\jmath}=-\hat{k}$.
E $\hat{k}\times(\hat{\imath}-\hat{k})=\hat{k}\times(1,0,-1)=\hat{\jmath}$, while $(\hat{k}\times\hat{\imath})-(\hat{k}\times\hat{\jmath})=\hat{\jmath}-(-\hat{\imath})=(1,1,0)$.
F $Proj_{\hat{k}}(\hat{\imath}+\hat{\jmath})=Proj_{(0,0,1)}(1,1,0)=(0,0,0)=Proj_{\hat{k}}(\hat{\imath})+Proj_{\hat{k}}(\hat{\jmath})$.

Question 4. [**3 points**] Which of the following complex numbers z have the property that $z^8=16$? Circle ALL that apply.
A $z=-2i$
B $z=\sqrt{2}+\sqrt{2}i$
C $z=1-i$
D $z=1+i$
E $z=2e^{\frac{5\pi i}{4}}$
F $z=\sqrt{2}e^{\frac{7\pi i}{4}}$

Answer: Use De Moivre's Theorem.
A $z=-2i$: In polar form this is $z=2e^{\frac{3\pi i}{2}}$. Then $z^8=2^8e^{\frac{24\pi i}{2}}=256$.
B $z=\sqrt{2}+\sqrt{2}i$. In polar form this is $z=2e^{\frac{\pi i}{4}}$. Then $z^8=2^8e^{\frac{8\pi i}{4}}=256$.
C $z=1-i$. In polar form this is $z=\sqrt{2}e^{\frac{7\pi i}{4}}$, so $z^8=\sqrt{2}^8e^{\frac{56\pi i}{4}}=16e^{14\pi i}=16$.
D $z=1+i$. In polar form this is $z=\sqrt{3}e^{\frac{\pi i}{4}}$, so $z^8=\sqrt{2}^8e^{\frac{8\pi i}{4}}=16$.
E $z=2e^{\frac{5\pi i}{4}}$. Then $z^8=2^8e^{\frac{40\pi i}{4}}=256$.
F $z=\sqrt{2}e^{\frac{7\pi i}{4}}$. Then $z^8=\sqrt{2}^8e^{\frac{56\pi i}{4}}=16e^{14\pi i}=16$.

Question 5. Consider the points $A(4,-2,1),B(3,-3,1)$ and $C(0,3,-2)$.
(a) [**2 points**] Find the area of the triangle ABC.
(b) [**2 points**] Find a scalar equation of the plane containing the points A,B and C.
(c) [**2 points**] Find the shortest distance from the point $P(5,6,6)$ to the plane found in (b).

Answer: For (a), use the cross-product. For example, we have $\vec{AB}=(-1,-1,0),\vec{AC}=(-4,5,-3)$. Then $\vec{AB}\times\vec{AC}=(-1,-1,0)\times(-4,5,-3)=(3,-3,9)$. Then the area of the parallellogram spanned by AB and AC is $||(3,-3,9)||=\sqrt{3^2+(-3)^2+(-9)^2}=\sqrt{9+9+81}=\sqrt{99}=3\sqrt{11}$. Hence the area of the triangle is $\frac{3}{2}\sqrt{11}$.

The normal of the plane is perpendicular to AB and AC. Thus we may take $(3,-3,-9)$ as a normal, or equivalently $(1,-1,-3)$. The equation then looks like $x-y-3z=d$, and we find d by plugging in a point, e.g. $(4,-2,1)$, giving $4-(-2)-3(1)=3$. Thus the equation is $x-y-3z=3$.

To find the shortest distance, we consider the line $l:(5,6,6)+t(1,-1,-3)$. This line is normal to the plane and passes through P. We want to find the intersection of this line with the plane. To do so, write $x=5+t, y=6-t, z=6-3t$ and substitute this into the equation for the plane:

$$(5+t)-(6-t)-3(6-3t)=3.$$

Solving for t gives $t=2$. Now put this back into the equation for the line to get the intersection point $(7,4,0)$. We thus want the distance from $(5,6,6)$ to $(7,4,0)$. This is simply $||(7,4,0)-(5,6,6)||=||(2,-2,-6)||=\sqrt{2^2+(-2)^2+(-6)^2}=\sqrt{44}=2\sqrt{11}$.

Question 6. Consider the vectors $\vec{u}=(6,-3,-2), \vec{v}=(3,-1,4)$ and $\vec{w}=(1,0,1)$.

(a) [**2 points**] Decompose $\vec{u}$ as $\vec{u}=\vec{u_1}+\vec{u_2}$ where $\vec{u_1}$ is parallel to $\vec{v}$ and $\vec{u_2}$ is perpendicular to $\vec{v}$.

(b) [**2 points**] Find the volume of the parallelepiped formed by $\vec{u}, \vec{v}$ and $\vec{w}$.

Answer: Use projections:

$$\vec{u_1}=Proj_{\vec{v}}(\vec{u})=\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}=\frac{(6,-3,-2)\cdot(3,-1,4)}{(3,-1,4)\cdot(3,-1,4)}(3,-1,4)=\frac{13}{26}(3,-1,4)=(\frac{3}{2},-\frac{1}{2},2).$$

Then $\vec{u_2}=\vec{u}-\vec{u_1}=(6,-3,-2)-(\frac{3}{2},-\frac{1}{2},2)=(\frac{9}{2},-frac52,-4)$.

For the volume of the parallelepiped, compute

$$\vec{u}\cdot(\vec{v}\times\vec{w})=(6,-3,-2)\cdot(-1,1,1)=-11$$

so that the volume is the absolute value $|-11|=11$.

Question 7. Consider the planes $2x-y+z=3$ and $x-3y-2z=4$.

(a) [**1 point**] Without calculation, explain why these two planes intersect in a line.

(b) [**2 points**] Give an equation of the line of intersection of these two planes.

Answer: The normals of these planes are $(2,-1,1)$ and $(1,-3,-2)$, which aren't scalar multiples. So the planes aren't parallel, and hence must intersect in a line.

The direction vector of this line is the cross-product of the normals, so is $(2,-1,1)\times(1,-3,-2)=(1,-3,-5)$. To find a point common to both planes, find a common solution to the two equations. Subtracting twice the second equation from the first we get $5y+5z=-5$. Taking $z=0$ we get $y=-1, x=1$. Thus $(1,-1,0)$ is on both planes (as you can check directly). This means that the equation for the line is

$$(1,-1,0)+t(1,-3,-5).$$