MAT 1339 B Winter 2011 February 11th, 10:00 Instructor S. Baek

Midterm #1

Max = 100

Name : ______________________________

Student Number: ______________________________

- Time: 80 min.
- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.
- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

Problem 1: (20 points) Let $f(x) = \begin{cases} \dfrac{\sqrt{x+2}-2}{x-2}, & \text{if } x > 2 \\ \sqrt{2-x}+C, & \text{if } x \le 2 \end{cases}$ for some constant C.

(a) (10 points) For what value of C is $f(x)$ continuous at $x = 2$?

(b) (10 points) Use the definition of the derivative to find the derivative of the function $g(x) = \sqrt{2-x} + C$.

Solution: *(a)*

$$\lim_{x\to 2^+} \frac{\sqrt{x+2}-2}{x-2} = \lim_{x\to 2^+} \left(\frac{\sqrt{x+2}-2}{x-2} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}\right) = \lim_{x\to 2^+} \frac{x-2}{x-2(\sqrt{x+2}+2)} = \frac{1}{4}.$$

On the other hand,

$$f(2) = \lim_{x\to 2^-} (\sqrt{2-x}+C) = C.$$

The function $f(x)$ is continuous at $x = 2$ if and only if $f(2) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, hence $C = \frac{1}{4}$.

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\sqrt{2-(x+h)}+C-(\sqrt{2-x}+C)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2-x-h}-\sqrt{2-x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2-x-h}-\sqrt{2-x}}{h} \cdot \frac{\sqrt{2-x-h}+\sqrt{2-x}}{\sqrt{2-x-h}+\sqrt{2-x}} \\ &= \lim_{h\to 0} \frac{-h}{h(\sqrt{2-x-h}+\sqrt{2-x})} \\ &= \frac{-1}{2\sqrt{2-x}}. \end{aligned}$$

Problem 2: (20 points)

(a) (10 points) Let $f(x) = \dfrac{(x^2+2x+3)^{20}}{2x^2+3}$. Find $f'(0)$.

(b) (10 points) Assume that $\lim_{x\to 2} \dfrac{(g(x)-1)^2}{x-2} = 3$. Find $\lim_{x\to 2} g(x)$.

Solution: (a)

$$f'(x) = \frac{20(x^2+2x+3)^{19}(2x+2)(2x^2+3) - (x^2+2x+3)^{20}(4x)}{(2x^2+3)^2}.$$

Therefore,

$$f'(0) = \frac{20 \cdot 3^{19} \cdot 2 \cdot 3}{3^2} = \frac{40 \cdot 3^{20}}{3^2} = 40 \cdot 3^{18}.$$

(b)

$$\lim_{x\to 2}(g(x)-1)^2 = \lim_{x\to 2}\{\frac{(g(x)-1)^2}{x-2} \cdot (x-2)\} = \lim_{x\to 2}\{\frac{(g(x)-1)^2}{x-2}\} \cdot \lim_{x\to 2}(x-2) = 3 \cdot 0 = 0.$$

Hence, $\lim_{x\to 2} g(x) = 1$.

Problem 3: (10 points)

Find the equation of the tangent line to the graph of $f(x) = (\frac{1}{4\sqrt{x+4}} + \frac{7}{8})(x^2+2)^4$ at $x = 0$.

Solution:

$$f'(x) = \frac{1}{4} \cdot (-\frac{1}{2}) \cdot \frac{1}{(x+4)^{\frac{3}{2}}} \cdot (x^2+2)^4 + (\frac{1}{4\sqrt{x+4}} + \frac{7}{8}) \cdot 4(x^2+2)^3 \cdot 2x.$$

Hence, $f'(0) = -\frac{1}{8} \cdot \frac{1}{4^{\frac{3}{2}}} \cdot 2^4 = -\frac{2^4}{2^3 \cdot 2^3} = -\frac{1}{4}$. As $f(0) = (\frac{1}{8} + \frac{7}{8})(2^4) = 2^4$, the equation of the tangent line is $y = -\frac{1}{4}x + 16$.

Problem 4: (20 points) Let $f(x) = x^4 - 4x^3 + 2$.

(a) (5 points) Calculate $f'(x)$ and find critical points.

(b) (5 points) Calculate $f''(x)$ and determine the intervals where $f(x)$ is concave up, concave down.

(c) (10 points) Sketch the graph of the function.

Solution: *(a)* $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$. Therefore, $f'(x) = 0$ has roots $x = 0$ and $x = 3$, which are critical points.

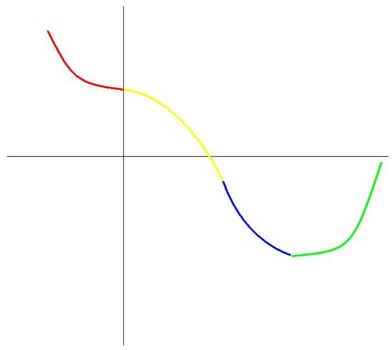
(b) $f''(x) = 12x^2 - 24x = 12x(x-2)$. Hence $f'(x) = 0$ has roots $x = 0$ and $x = 2$. We have the following table:

Interval	$(-\infty,0)$	$(0,2)$	$(2,3)$	$(3,\infty)$
Sign of $f'(x)$	$-$	$-$	$-$	$+$
Sign of $f''(x)$	$+$	$-$	$+$	$+$
Shape of curve	DU(Red)	DD(Yellow)	DU(Blue)	IU(Green)

,where DU:=Decreasing and concave Up, IU:=Increasing and concave Up, and DD:=Decreasing and concave Down.

According to the table, the interval where $f(x)$ is concave up is $(-\infty, 0) \cup (2,3) \cup (3,\infty)$ and the interval where $f(x)$ is concave down is $(0,2)$.

(c)



Problem 5: (30 points) Let $f(x) = \dfrac{1}{x^2 - 2x}$.

(a) (5 points) Determine the domain of the function.

(b) (5 points) Find the vertical asymptotes.

(c) (5 points) Calculate $f'(x)$ and find critical points.

(d) (5 points) Calculate $f''(x)$ and determine the intervals where $f(x)$ is concave up, concave down.

(e) (10 points) Sketch the graph of the function.

Solution: (a) $\mathbb{R}\backslash\{0, 2\}$.

(b) As $\lim_{x\to 0^+} \dfrac{1}{x(x-2)} = -\infty$ (or $\lim_{x\to 0^-} \dfrac{1}{x(x-2)} = \infty$), $x = 0$ is a vertical asymptote. Similarly, as $\lim_{x\to 2^+} \dfrac{1}{x(x-2)} = \infty$ (or $\lim_{x\to 2^-} \dfrac{1}{x(x-2)} = -\infty$), $x = 2$ is a vertical asymptote.

(c) $f'(x) = \dfrac{-2x+2}{(x^2-2x)^2}$. Therefore, the root $x = 1$ of $f'(x) = 0$ is the critical point.

(d) $f''(x) = \dfrac{-2(x^2-2x)^2 - (-2x+2)\cdot 2\cdot (x^2-2x)(2x-2)}{(x^2-2x)^4} = \dfrac{-2(x^2-2x)(-3x^2+6x-4)}{(x^2-2x)^4}$.
As $-3x^2+6x-4 = 0$ has no roots (the discriminant is negative), $f''(x) = 0$ has no roots.

We have the following table:

Interval	$(-\infty$,0)	(0,1)	(1,2)	(2,∞)
Sign of $f'(x)$	+	+	−	−
Sign of $f''(x)$	+	−	−	+
Shape of curve	IU(Blue)	ID(Red)	DD(Green)	DU(Yellow)

,where DU:=Decreasing and concave Up, IU:=Increasing and concave Up, ID:=Increasing and concave Down, and DD:=Decreasing and concave Down.

According to the table, the interval where $f(x)$ is concave up is $(-\infty, 0) \cup (2, \infty)$ and the interval where $f(x)$ is concave down is $(0, 2)$.

Space for problem 5