COMP 1805 – Discrete Structures I
Assignment 4

Due December 2 by 5:00pm

Place your assignment in the School of Computer Science Drop Boxes in HP 3115.
Be sure to use the box for this course.

Write down your name and student number on *every* page. The questions *must* be answered in order and your assignment sheets *must* be stapled. All questions (or subquestions) will be marked out of 2: 2 points will be awarded for a correct answer, 1 point will be awarded for a partially correct answer (one major detail or a few minor details missing or wrong), and 0 points will be awarded for a completely incorrect answer.

1. Consider the following relations defined on the integers. Determine if each relation is reflexive, symmetric, anti-symmetric, and transitive.

   (a) $\{(a, b) \mid a \neq b\}$

   Not reflexive since $(1, 1)$ is not in the relation (since it is not true that $1 \neq 1$). Symmetric since if $a \neq b$ then $b \neq a$ for any integers a, b. Not antisymmetric since $(0, 1)$ and $(1, 0)$ are both in the relation, but $0 \neq 1$. Not transitive since $(0, 1)$ and $(1, 0)$ are in the relation, but $(0, 0)$ is not.

   (b) $\{(a, b) \mid a = 4b\}$

   Not reflexive since $(1, 1)$ is not in the relation. Not symmetric since $(4, 1)$ is in the relation but $(1, 4)$ is not. The relation is antisymmetric. Suppose (x, y) and (y, x) are in the relation, and assume (for the sake of contradiction) that $x \neq y$. We have $x = 4y$, which means that $y = x/4$. However, we also have that $y = 4x$. Therefore, $x/4 = 4x$, which is a contradiction unless $x = 0$ (in which case $y = x = 0$). The relation is not transitive since $(16, 4)$ and $(4, 1)$ are in the relation, but $(16, 1)$ is not.

   (c) $\{(a, b) \mid a/b \geq 0\}$

   Not reflexive since $(0, 0)$ is not in the relation ($0/0$ is not defined). Not symmetric since $(0, 1)$ is in the relation but $(1, 0)$ is not in the relation (since $1/0$ is not defined). Not antisymmetric, since $(1, 2)$ and $(2, 1)$ are both in the relation, but $1 \neq 2$. The relation is transitive. Suppose we have (a, b) and (b, c) in the relation. Since (a, b) is in the relation, $b \neq 0$. If $b > 0$, then $a > 0$ and $c > 0$ and so $a/c \geq 0$. Otherwise, if $b < 0$, then $a < 0$ and $c < 0$, and so $a/c \geq 0$.

   (d) $\{(a, b) \mid a > b^3\}$

   Not antisymmetric: $(-4,-3),(-3,-4)$
   Not transitive: $(-4,-3),(-3,-4)$, but not $(-4,-4)$

   Not reflexive since $(1, 1)$ is not in the relation since it is not true that $1 > 1^3 = 1$. Not symmetric since $(2, 1)$ is in the relation (since $2 > 1^3 = 1$) but $(1, 2)$ is not in the relation (since $1 \leq 2^3 = 8$). The relation is ~~antisymmetric. Suppose (x, y) and (y, x) are in the relation, and assume (for the sake of contradiction) that $x \neq y$. We have $x > y^3$ and $y > x^3$. Therefore, $x > x^6$, which is not true for any integer x and is therefore a contradicton. The relation is transitive. Suppose (x, y) and (y, z) are in the relation. Then $x > y^3$ and $y > z^3$. Then $x > (z^3)^3 = z^9$. Since $z^9 > z^3$, it follows that $x > z^3$ and so (x, z) is in the relation.~~

   (e) $\{(a, b) \mid a > 3b + 4b^2\}$

   Not reflexive since $(1, 1)$ is not in the relation since it is not true that $1 > 3(1) + 4(1)^2 = 7$. Not symmetric since $(8, 1)$ is in the relation (since $8 > 3(1) + 4(1)^2 = 7$), but $(1, 8)$ is not in the relation (since it is not true that $1 > 3(8) + 4(8)^2 = 280$). The relation is antisymmetric. Suppose (x, y) and (y, x) are in the relation. Then $x > 3y + 4y^2$ and $y > 3x + 3x^2$. Substituting, we have $x > 3y + 4y^2 > 3(3x + 4x^2) + 4(3x + 4x^2)^2 = 64x^4 + 96x^3 + 18x^2 + 9x$, which is not true for any integer x and is therefore a contradiction. The relation is transitive. Suppose (x, y) and (y, z) are in the relation. Then $x > 3y + 4y^2$ and $y > 3z + 4z^2$. Substituting, we have $x > 3y + 4y^z > 3(3z + 4z^2) + 4(3z + 4z^2)^2 = 64z^4 + 96z^3 + 18z^2 + 9z = 4z + 4z^2 + (64z^4 + 96z^3 + 14z^2 + 5z)$, and so (x, z) is in the relation.

2. Let S and T be transitive relations defined on a set A. Determine the following:

   (a) Is $S \cap T$ transitive?

   Yes. Suppose $(x, y) \in S \cap T$ and $(y, z) \in S \cap T$. Then, by the definition of intersection, $(x, y) \in S$ and $(x, y) \in T$, and similarly $y, z \in S$ and $y, z \in T$. Since S and T are transitive, we have that $(x, z) \in S$ and $(x, z) \in T$. Therefore, by the definition of intersection, $(x, z) \in S \cap T$.

   (b) Is $S \cup T$ transitive?

   No. Let $A = \{1, 2, 3, 4\}$ and let $S = \{(1, 2), (2, 3), (1, 3)\}$ and $T = \{(2, 3), (3, 4), (2, 4)\}$. Notice that both S and T are transitive. Now, $(1, 2) \in S \cup T$ and $(2, 4) \in S \cup T$, but $(1, 4)$ is not in $S \cup T$ and therefore $S \cup T$ is not transitive.

   (c) Is $S \circ T$ transitive?

   No. Let $A = \{1, 2, 3, 4, 5\}$ and let $S = \{(4, 2), (5, 3)\}$ and $T = \{(1, 4), (2, 5)\}$. Notice that both S and T are transitive. However, $S \circ T = \{(1, 2), (2, 3)\}$. Since $(1, 3)$ is not in $S \circ T$, it is not transitive.

3. Let $A = \{a, b, c, d\}$ and let $R = \{(a, a), (b, a), (b, c), (c, a), (c, c), (c, d), (d, a), (d, c)\}$ be a relation on A. Find the transitive closure of R.

   The matrix representation of R is

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

   We have:

$$M_{R^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad M_{R^3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad M_{R^4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

   Computing $M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$ gives

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

   which means the transitive closure of R is $R \cup \{(b, d), (d, d)\}$. It is also possible to do this by inspection.

4. Let A be the set of all ordered pairs of non-negative integers. This means that $A = \{(a, b) \mid a, b \geq 0\}$. Let R be a relation defined on A such that $R = \{((a, b), (c, d)) \mid a + d = b + c\}$. Determine whether or not R is an equivalence relation.

   Yes, this is an equivalence relation.

   - Reflexivity: Is $((a, b), (a, b)) \in R$? We have that $a + b = b + a$, therefore the relation is reflexive.

- Symmetry: Suppose $((a,b),(c,d)) \in R$. Then $a+d=b+c$. Rearranging, this means that $c+b=d+a$, and so $((c,d),(a,b)) \in R$. Therefore, the relation is symmetric.
- Transitivity: Suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. We want to show that $((a,b),(e,f)) \in R$. Since $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$, we have $a+d=b+c$ and $c+f=d+e$. Adding these together gives us $a+d+c+f=b+c+d+e$. Subtracting $d+c$ from both sides gives us $a+f=b+c$, which means that $((a,b),(e,f)) \in R$ and so the relation is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

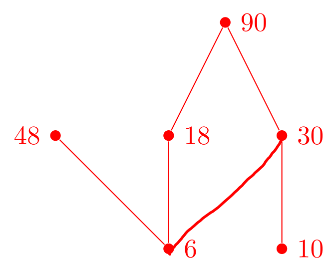
5. Let $A=\{6,10,18,30,48,90\}$ and let $R=\{(a,b) \mid a \text{ divides } b\}$.

   (a) Show that R is a partial order.

   - Reflexivity: $a \mid a$ for any integer a since $a = 1 \times a$.
   - Antisymmetry: If $a \mid b$, then $b = k_1 a$ for some integer k_1. Similarly, if $b \mid a$, then $a = k_2 b$ for some integer k_2. Therefore, $b = k_1 a = k_1(k_2 b)$, and $b = k_1 k_2 b$ means $k_1 k_2 = 1$. Since k_1 and k_2 are integers, the only way this can happen is if $k_1 = k_2 = 1$.
   - Transitivity: If $a \mid b$, then $b = k_1 a$ for some integer k_1. Similarly, if $b \mid c$, then $c = k_2 b$ for some integer k_2. We therefore have $c = k_2 b = k_2(k_1 a) = (k_2 k_1)a$, which shows that $a \mid c$.

   Since R is reflexive, antisymmetric, and transitive, it is a partial order.

   (b) Draw the Hasse diagram for R.



   (c) Find the maximal elements of R.

   $48, 90$

   (d) Find the minimal elements of R.

   $6, 10$

   (e) Find the least upper bound of $\{6, 10\}$ or explain why one does not exist.

   ~~90~~ 30

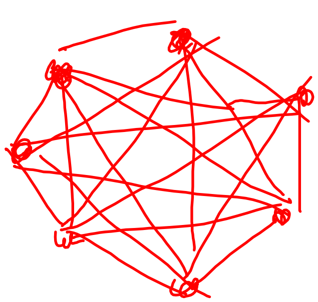
   (f) Find the least element or explain why one does not exist.

   Two minimal elements $(6, 10)$, therefore no least element.

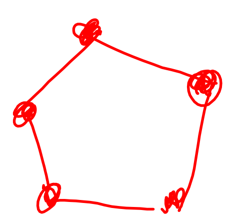
6. Draw the following graphs:

   (a) K_7 (b) C_5 (c) W_8 (d) $K_{7,4}$

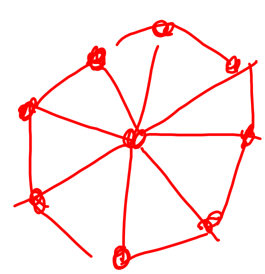
(a)



(b)



(c)



(d)