COMP 1805 – Discrete Structures I
Assignment 3

Due November 11 by 5:00pm

Place your assignment in the School of Computer Science Drop Boxes in HP 3115.
Be sure to use the box for this course.

Write down your name and student number on *every* page. The questions *must* be answered in order and your assignment sheets *must* be stapled. All questions (or subquestions) will be marked out of 2: 2 points will be awarded for a correct answer, 1 point will be awarded for a partially correct answer (one major detail or a few minor details missing or wrong), and 0 points will be awarded for a completely incorrect answer.

1. Compute closed forms for the following summations.

   (a) $\sum_{j=32}^{76} 3 = (76 - 32 + 1) \times 3 = 45 \times 3 = 135$

   (b) $\sum_{i=t}^{n} 3 = (n - t + 1) \times 3$

   (c) $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{n} (k+1)$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{n} (k+1) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i} \left(\sum_{k=1}^{n} (k+1) - \sum_{k=1}^{j-1} (k+1) \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i} \left(\sum_{k=2}^{n+1} k - \sum_{k=2}^{j} k \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i} \left(\frac{(n+1)(n+2)}{2} - 1 - \frac{j(j+1)}{2} + 1 \right) \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} (n^2 + 3n + 2 - j^2 - j) \\
&= \frac{1}{2} \sum_{i=1}^{n} \left(\sum_{j=1}^{i} (n^2 + 3n + 2) - \sum_{j=1}^{i} j^2 - \sum_{j=1}^{i} j \right) \\
&= \frac{1}{2} \sum_{i=1}^{n} \left(i(n^2 + 3n + 2) - \frac{i(i+1)(2i+1)}{6} - \frac{i(i+1)}{2} \right) \\
&= \frac{1}{2} \sum_{i=1}^{n} \left(i(n^2 + 3n + 2) - \frac{1}{6} \left(2i^3 + 3i^3 + i\right) - \frac{1}{2} \left(i^2 + i\right) \right) \\
&= \frac{1}{2} \sum_{i=1}^{n} \left(i(n^2 + 3n + 2) - \frac{1}{3} i^3 - \frac{1}{2} i^2 - \frac{1}{6} i - \frac{1}{2} i^2 - \frac{1}{2} i \right)
\end{aligned}
$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(i(n^2+3n+2) - \frac{1}{3}i^3 - i^2 - \frac{2}{3}i\right)$$
$$= \frac{n^2+3n+2}{2}\sum_{i=1}^{n} i - \frac{1}{6}\sum_{i=1}^{n} i^3 - \frac{1}{2}\sum_{i=1}^{n} i^2 - \frac{1}{3}\sum_{i=1}^{n} i$$
$$= \frac{(n^2+3n+2)}{2}\frac{n(n+1)}{2} - \frac{1}{6}\frac{n^2(n+1)^2}{4} - \frac{1}{2}\frac{n(n+1)(2n+1)}{6} - \frac{1}{3}\frac{n(n+1)}{2}$$
$$= \frac{5}{24}n^4 + \frac{3}{4}n^3 + \frac{19}{24}n^2 + \frac{1}{4}n$$

Note: you may stop at the second-to-last line; there is no need to simplify further once the Σs are gone.

(d) $\sum_{k=1}^{n}(3k-2) = \sum_{k=1}^{n} 3k - \sum_{k=1}^{n} 2 = \frac{3}{2}n(n+1) - 2n$

2. Let $f(n) = \sum_{i=1}^{n}\sum_{j=i}^{n} 4j$. Determine a closed form for this summation and determine its order of growth.

$$\begin{aligned}
f(n) &= \sum_{i=1}^{n}\sum_{j=i}^{n} 4j \\
&= 4\sum_{i=1}^{n}\sum_{j=i}^{n} j \\
&= 4\sum_{i=1}^{n}\left(\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j\right) \\
&= 4\sum_{i=1}^{n}\left(\frac{n(n+1)}{2} - \sum i(i-1)2\right) \\
&= 2\sum_{i=1}^{n}(n^2+n-i^2+i) \\
&= 2\sum_{i=1}^{n}(n^2+n) - 2\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i \\
&= 2n(n^2+n) - 2\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} \\
&= 2n(n^2+n) - \frac{n(n+1)(2n+1)}{3} + n(n+1) \\
&= \frac{4}{3}n^3 + 2n^2 + \frac{2}{3}n
\end{aligned}$$

Now, you should recognize that $f(n)$ is $O(n^3)$:

$$\begin{aligned}
f(n) &= \frac{4}{3}n^3 + 2n^2 + \frac{2}{3}n \\
&\le \frac{4}{3}n^3 + 2n^3 + \frac{2}{3}n^3 \\
&= \left(\frac{4}{3} + 2 + \frac{2}{3}\right)n^3 \\
&= 4n^3
\end{aligned}$$

Thus, we take $c = 4$. Since each step is true for all positive n, we can take $k = 1$. Therefore, $f(n)$ is $O(n^3)$.

3. Determine if $f(n)$ is $O(g(n))$ or $f(n)$ is $\Omega(g(n))$ or $f(n)$ is $\Theta(g(n))$.

   (a) $f(n) = 2n^2 - 4n + 2, g(n) = n^2$

   First note that $f(n)$ is $\Theta(g(n))$ since the leading exponents in both polynomials match. Now, to show this we have to show that it is both $O(g(n))$ and $\Omega(g(n))$. First, we show $f(n)$ is $O(g(n))$:

$$\begin{aligned} f(n) &= 2n^2 - 4n + 2 \\ &\leq n^2 + 2 \text{ when } n > 0 \\ &\leq n^2 + 2n^2 \text{ when } n \geq 1 \\ &= 3n^2 \end{aligned}$$

   Therefore, $f(n) \leq 5g(n)$ for $n \geq 1$ and so $f(n)$ is $O(g(n))$ with $c = 5, k = 1$. Now, to show $f(n)$ is $\Omega(g(n))$:

$$\begin{aligned} f(n) &= 2n^2 - 4n + 2 \\ &\geq 2n^2 - 4n \\ &= n^2/2 + (n^2 - 4n) \\ &\geq n^2/2 \text{ when } n \geq 4 \text{ since that is when } n^2 - 4n \geq 0 \end{aligned}$$

   Therefore, $f(n) \geq g(n)/2$ for $n \geq 4$ and so $f(n)$ is $\Omega(g(n))$ with $c = 1/2, k = 4$. Furthermore, since $f(n)$ is $O(g(n))$ and $\Omega(g(n))$, we have that $f(n)$ is $\Theta(g(n))$.

   (b) $f(n) = 5n^4 - 3n + 2, g(n) = n^3$

   In this case you should recognize that $f(n)$ is not $O(g(n))$ since the exponent in $f(n)$ is 4 and in $g(n)$ is 3. To prove this, suppose that $f(n)$ were in fact $O(g(n))$. Then:

$$\begin{aligned} 5n^4 - 3n + 2 &\leq cn^3, \forall n \geq k \\ 5n - \frac{3}{n^2} + \frac{2}{n^2} &\leq c, \forall n \geq k \text{ divide both sides by } n^3 \end{aligned}$$

   Note that the left side of the equation can never be smaller than a fixed constant c since it is a linear function of n. Therefore, $f(n)$ is not $O(n^3)$ (and therefore not $\Theta(n^3)$, either).

   However, $f(n)$ is $\Omega(g(n))$. To see this:

$$\begin{aligned} f(n) &= 5n^4 - 3n + 2 \\ &\geq 5n^4 - 3n \\ &= 4n^4 + (n^4 - 3n) \\ &\geq 4n^4 \text{ when } n \geq 2 \text{ since that is when } n^4 - 3n \geq 0 \\ &\geq 4n^3 \end{aligned}$$

   Therefore, $f(n) \geq 4g(n)$ for $n \geq 2$ and so $f(n)$ is $\Omega(g(n))$ with $c = 4, k = 2$.

   (c) $f(n) = 4n^4 + 5n^3 - 12n + 8, g(n) = n^5$

   In this case you should recognize that $f(n)$ is not $\Omega(g(n))$ since the exponent in $f(n)$ is 4 and in $g(n)$ is 5. To prove this, suppose that $f(n)$ were in fact $\Omega(g(n))$. Then:

$$\begin{aligned} 4n^4 + 5n^3 - 12n + 8 &\geq cn^5, \forall n \geq k \\ \frac{4}{n} + \frac{5}{n^2} - \frac{12}{n^4} + \frac{8}{n^5} &\geq c, \forall n \geq k \text{ divide both sides by } n^5 \end{aligned}$$

   Note that the left side of the equation decreases toward zero as n gets bigger, so it can never be larger than any fixed constant c. Therefore, $f(n)$ is not $\Omega(n^5)$ (and therefore not $\Theta(n^5)$, either).

However, $f(n)$ is $O(g(n))$. To see this:

$$\begin{aligned} f(n) &= 4n^4 + 5n^3 - 12n + 8 \\ &\leq 4n^4 + 5n^3 + 8 \\ &\leq 4n^4 + 5n^4 + 8n^4 \text{ when } n \geq 1 \\ &= 17n^4 \end{aligned}$$

Therefore, $f(n) \geq 17g(n)$ for $n \geq 1$ and so $f(n)$ is $\Omega(g(n))$ with $c = 17, k = 1$.

4. If $f(n)$ is $O(g(n))$ and $g(n)$ is $\Omega(h(n))$, determine if the following statements are true.

   (a) $f(n)$ is $O(h(n))$

   False. Let $f(n) = n^2$, $g(n) = n^2$ and $h(n) = n$. We know n^2 is $O(n^2)$ (take $c = k = 1$) and we know n^2 is $\Omega(n)$ (take $c = k = 1$). However, n^2 is not $O(n)$: if $n^2 \leq cn$ for $n > k$, then we divide through by n to see that $n \leq c$ for $n > k$. This says that n is less than some fixed constant, which cannot be the case.

   (b) $f(n)$ is $\Omega(h(n))$

   False. Let $f(n) = n$, $g(n) = n^2$ and $h(n) = n^2$. We know n is $O(n^2)$ (take $c = k = 1$) and we know n^2 is $\Omega(n^2)$ (take $c = k = 1$). However, n is not $\Omega(n^2)$: if $n \geq cn^2$ for $n > k$, then we can divide through by n^2 to see that $1/n \geq c$ for $n > k$. However, as n grows larger, $1/n$ becomes smaller, and so it cannot always be greater than some fixed constant.

   (c) $f(n)$ is $\Omega(g(n))$

   False. The same counterexample and analysis as the previous question applies: n is not $\Omega(n^2)$.

   (d) $g(n)$ is $\Omega(f(n))$

   True. Since $f(n)$ is $O(g(n))$, we know that $f(n) \leq cg(n)$ for $n > k$. Thus, $g(n) \geq (1/c)f(n)$ for $n > k$, so $g(n)$ is $\Omega(f(n))$.

   (e) $h(n)$ is $O(g(n))$

   True. Since $g(n)$ is $\Omega(h(n))$, we know that $g(n) \geq ch(n)$ for $n > k$. Thus, $h(n) \leq (1/c)g(n)$ for $n > k$, so $h(n)$ is $O(g(n))$.

5. Prove that $4^n < n!$ for all integers $n \geq 9$.

   Basis step ($n = 9$): $4^9 = 262144 < 362880 = 9!$.

   Inductive hypothesis: $4^k < k!$.

   Inductive step: We must show $4^{k+1} < (k+1)!$.

$$\begin{aligned} 4^{k+1} &= 4 \times 4^k \\ &< 4 \times k! \text{ by inductive hypothesis} \\ &< (k+1) \times k! \text{ since } k > 9 \\ &= (k+1)! \text{ by definition of factorial} \end{aligned}$$

6. Prove that $\sum_{i=1}^{n} 3/4^i < 4$ for all integers $n \geq 2$.

   Basis step ($n = 2$): $\sum_{i=1}^{n} 3/4^i = 3/4 + 3/16 = 15/16 < 4$

   Inductive hypothesis: $\sum_{i=1}^{k} 3/4^i < 4$

   Inductive step: We must show $\sum_{i=1}^{k+1} 3/4^i < 4$. Notice that $\sum_{i=1}^{k} 3/4^i = 3\sum_{i=1}^{k} 1/4^i = 3\frac{\frac{1}{4}^{k+1}-1}{\frac{1}{4}-1}$. Similarly, $\sum_{i=1}^{k+1} 3/4^i = 3\sum_{i=1}^{k+1} 1/4^i = 3\frac{\frac{1}{4}^{k+2}-1}{\frac{1}{4}-1}$.

We have:

$$\begin{aligned}
\sum_{i=1}^{k+1} 3/4^i &= 3\frac{\frac{1}{4}^{k+2}-1}{\frac{1}{4}-1} \\
&= 3\frac{\left(\frac{1}{4}^{k+1}-1\right)\left(\frac{1}{4}\right)+\frac{1}{4}-1}{\frac{1}{4}-1} \\
&= 3\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}^{k+1}-1\right)-\frac{3}{4}}{\frac{1}{4}-1} \\
&= 3\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}^{k+1}-1\right)}{\frac{1}{4}-1}-3\frac{\frac{3}{4}}{\frac{1}{4}-1} \\
&= \left(\frac{1}{4}\right)3\frac{\frac{1}{4}^{k+1}-1}{\frac{1}{4}-1}-3\frac{\frac{3}{4}}{\frac{1}{4}-1} \\
&< \left(\frac{1}{4}\right)4-3\frac{\frac{3}{4}}{\frac{1}{4}-1} \text{ by inductive hypothesis} \\
&= 1+3\times 1 \\
&= 4
\end{aligned}$$