

COMP 1805 – Discrete Structures I

Assignment 2

Due October 14 by 5:00pm

Place your assignment in the School of Computer Science Drop Boxes in HP 3115.
Be sure to use the box for this course.

Write down your name and student number on *every* page. The questions *must* be answered in order and your assignment sheets *must* be stapled. All questions (or subquestions) will be marked out of 2: 2 points will be awarded for a correct answer, 1 point will be awarded for a partially correct answer (one major detail or a few minor details missing or wrong), and 0 points will be awarded for a completely incorrect answer.

1. Prove that if m and n are integers and mn is even, then either m is even or n is even.

Solution: Proof by contradiction. Assume that m and n are integers and mn is even, but neither m nor n is even. Therefore, m and n are both odd. Thus, $n = 2k_1 + 1$ and $m = 2k_2 + 1$. (Note that we use k_1 and k_2 because we don't know if $m = n$ or not.) Now, $mn = (2k_1 + 1)(2k_2 + 1) = 4k_1k_2 + 2k_1 + 2k_2 + 1 = 2(2k_1k_2 + k_1 + k_2) + 1$, which is odd since $2k_1k_2 + k_1 + k_2$ is an integer. However, we assumed that mn is even. This is a contradiction, so our assumption must be wrong. Thus, m is even or n is even.

2. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$.

Solution: Proof by cases. Consider two cases: either $x \leq y$ or $x > y$. (Note that for any x and y , precisely one of these two statements is true.)

- if $x \leq y$, then $\max(x, y) = y$ and $\min(x, y) = x$, and so $\max(x, y) + \min(x, y) = y + x = x + y$
- if $x > y$, then $\max(x, y) = x$ and $\min(x, y) = y$, and so $\max(x, y) + \min(x, y) = x + y$

3. Prove that if a , b and c are real numbers with $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

Solution: We first prove that a solution exists. If $ax + b = c$, then $x = (c - b)/a$. Note that we know that this is valid because $a \neq 0$. To prove uniqueness, assume that we have two solutions x and y . Then $ax + b = c$ and $ay + b = c$. Thus, $ax + b = ay + b$, so that $ax = ay$, so that $x = y$.

4. List explicitly the members of the following sets.

(a) $\{x \mid x \text{ is a negative integer and } x^2 < 70\}$

Solution: $\{-1, -2, -3, -4, -5, -6, -7, -8\}$

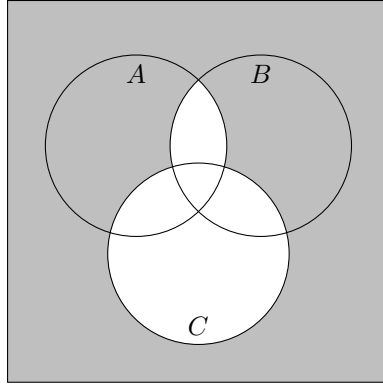
(b) $\{x \mid x \text{ is the cube of an integer and } 0 \leq x < 152\}$

Solution: $\{0, 1, 8, 27, 64, 125\}$

5. Let A , B and C be sets. Draw the Venn diagram for the following sets.

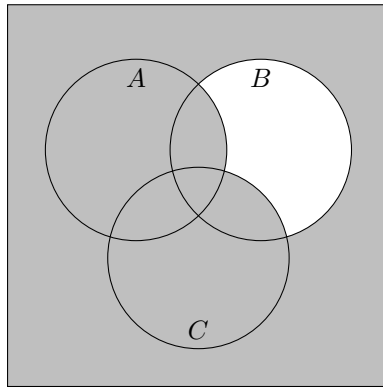
(a) $\overline{(A \cap B)} \cap \overline{C}$

Solution:



(b) $A \cup \overline{(B \setminus C)}$

Solution:



6. Let A , B and C be sets. Determine if the following pairs of sets are equivalent using set identities.

(a) $(A \setminus B)$ and $(A \setminus C) \setminus (B \setminus C)$

Solution: Apply the following identities, starting with $(A \setminus C) \setminus (B \setminus C)$.

$$\begin{aligned}
 & (A \setminus C) \setminus (B \setminus C) \\
 = & (A \cap \overline{C}) \cap \overline{(B \cap \overline{C})} && \text{Difference Equivalence} \\
 = & (A \cap \overline{C}) \cap (\overline{B} \cup C) && \text{De Morgan's Law} \\
 = & A \cap ((\overline{C} \cap \overline{B}) \cup (\overline{C} \cap C)) && \text{Distributive Law} \\
 = & A \cap ((\overline{C} \cap \overline{B}) \cup \emptyset) && \text{Complement Law} \\
 = & A \cap (\overline{C} \cap \overline{B}) && \text{Identity Law} \\
 = & A \cap \overline{B} \cap \overline{C} && \text{Associative, Commutative Laws} \\
 = & (A \setminus B) \cap \overline{C} && \text{Associative, Commutative Laws}
 \end{aligned}$$

If we let $\overline{C} = \emptyset$ (so that $C = U$), then $(A \setminus B) \cap \overline{C} = \emptyset$. However, this does not restrict our choice of A and B . So let $A = \{1, 2\}$, $B = \{1\}$ and $C = \emptyset$. Then $A \setminus B = \{2\}$, while $(A \setminus C) \setminus (B \setminus C) = (A \setminus B) \cap \overline{C} = \emptyset$. Therefore, the sets are not equivalent.

(b) $(A \cap B) \cup (A \cap \overline{B})$ and \overline{A}

Solution: Apply the following identities, starting with $(A \cap B) \cup (A \cap \overline{B})$.

$$\begin{aligned}
 & (A \cap B) \cup (A \cap \overline{B}) \\
 = & A \cap (B \cup \overline{B}) && \text{Distributive Law} \\
 = & A \cap U && \text{Complement Law} \\
 = & A && \text{Identity Law}
 \end{aligned}$$

It is clear that $A \neq \overline{A}$, so the two sets are not equal.

7. Let A , B and C be sets. Determine if the following pairs of sets are equivalent using membership tables.

(a) $(A \setminus B) \cup (B \setminus A)$ and $\overline{(A \cup B)}$

Solution:

A	B	$(A \setminus B) \cup (B \setminus A)$	$\overline{(A \cup B)}$
1	1	0	0
1	0	1	0
0	1	1	0
0	0	0	1

Since the columns do not match, the sets are not equal.

(b) $\overline{(C \setminus A) \cap (B \setminus A)}$ and $\overline{(C \cap B) \cap \overline{A}}$

Solution:

A	B	C	$\overline{(C \setminus A) \cap (B \setminus A)}$	$\overline{(C \cap B) \cap \overline{A}}$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	1	1
0	1	1	0	0
0	1	0	1	1
0	0	1	1	1
0	0	0	1	1

Since the columns match, the sets are equal.

8. Determine whether or not the following functions from the real numbers to the real numbers are injective, surjective and bijective.

(a) $f(x) = 2|x| + 3$

Solution: Not injective since $f(1) = f(-1) = 5$, but $1 \neq -1$. Not surjective since there is no x such that $f(x) = -1$, since $f(x) \geq 3$ for all x . Not bijective since neither injective nor surjective.

(b) $f(x) = 3.1415926535897932384626$

Solution: Not injective since $f(2) = f(3) = 3.1415926535897932384626$, but $2 \neq 3$. Not surjective since there is no x of such that $f(x) = 1.234$. Not bijective since neither injective nor surjective.

(c) $f(x) = 4x^3 + 3$

Solution: This function is injective: suppose $f(x) = f(y)$. Then $4x^3 + 3 = 4y^3 + 3$, so $4x^3 = 4y^3$, so $x^3 = y^3$, and so $x = y$. This function is surjective: if we want output z , give input $\sqrt[3]{(z-2)/3}$. Since the function is both injective and surjective, it is bijective.

9. Let f be a surjective function from A to B , h be an injective function from A to B , g be a surjective function from B to C , and k be a surjective function from B to C . Determine whether or not each of the following functions are injective, surjective, and bijective.

(a) $f \circ g$

Solution: This does not define a function, since the codomain of g is C , but the domain of f is A .

(b) $g \circ f$

Solution: This defines a function from A to C .

The function is not necessarily injective. To see this, take $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1, 2\}$ with the functions $f(1) = 1$, $f(2) = 1$, $f(3) = 2$ and $g(1) = 1$, $g(2) = 2$. It is easy to see that both f and g are surjective, as required. However, $f(g(1)) = f(1) = 1$ and $f(g(2)) = f(2) = 1$, but $1 \neq 2$.

This function is surjective. To show this, we need to show that for every element $x \in C$, there exists at least one element $y \in A$ such that $g(f(x)) = y$. Let z be an arbitrary element of C . Since g is surjective, there is at least one element $b \in B$ such that $g(b) = z$. Since f is surjective, there is at least one element $a \in A$ such that $f(a) = b$. This means that $g(f(a)) = g(b) = z$. Since this is true for any element $x \in C$, it follows that function is surjective.

(c) $g \circ h$

Solution: This defines a function from A to C .

The function is not necessarily injective. To see this, take $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{1\}$ with the functions $h(1) = 1$, $h(2) = 2$ and $g(1) = 1$, $g(2) = 1$. It is easy to see that h is injective and g is surjective. However, $g(h(1)) = g(1) = 1$ and $g(h(2)) = g(2) = 1$, but $1 \neq 2$.

The function is not necessarily surjective. To see this, take $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3\}$ with the functions $h(1) = 1$, $h(2) = 2$ and $g(1) = 1$, $g(2) = 2$, $g(3) = 3$. It is easy to see that h is injective and g is surjective. However, there is no pre-image x such that $g(h(x)) = 3$.

(d) $k \circ f$

Solution: Same as (b).