COMP 1805 – Discrete Structures
Assignment 1

Due September 23 by 5:00pm

Place your assignment in the School of Computer Science Drop Boxes in HP 3115.
Be sure to use the box for this course.

Write down your name and student number on *every* page. The questions *must* be answered in order and your assignment sheets *must* be stapled. All questions (or subquestions) will be marked out of 2: 2 points will be awarded for a correct answer, 1 point will be awarded for a partially correct answer (one major detail or a few minor details missing or wrong), and 0 points will be awarded for a completely incorrect answer.

1. Is the statement "Always attend class" a proposition? Justify your answer.

   Solution: A proposition is a declarative sentence. The sentence "Always attend class" is imperative (not declarative) in nature; hence it is not a proposition. An imperative sentence is neither true nor false.

2. Translate the following logical expressions into English, where a is the statement "Google is a good search engine" and b is the statement "The course webpage is found."

   (a) $\neg a \wedge \neg b$

   Solution: Google is not a good search engine and the course webpage is not found.

   (b) $\neg b \rightarrow \neg a$

   Solution: If the course webpage is not found, then Google is not a good search engine.

   (c) $b \vee \neg a$

   Solution: The course webpage is found or Google is not a good search engine.

3. Translate the following English statements into logical expressions.

   (a) If the hard drive crashes, then the data is lost.

   Solution: Let a be the proposition "The hard drive crashes" and b be the proposition "The data is lost." The statement is $a \rightarrow b$.

   (b) The circuit board is overheating and the mouse has two buttons.

   Solution: Let a be the proposition "The circuit board is overheating" and b be the proposition "The mouse has two buttons." The statement is $a \wedge b$.

   (c) I tried to debug the program, but it crashes anyway.

   Solution: Let a be the proposition "I debug the program" and b be the proposition "The program crashes." The statement is $a \wedge b$. (Recall that "but" indicates conjunction.)

4. State (in English) the inverse, converse and contrapositive of the implication "If I study hard, then I will pass the course."

   Solution: Let a be the proposition "I study hard" and b be the proposition "I will pass the course." The given statement is translated into the logical expression $a \rightarrow b$.

- The inverse of $a \to b$ is $\neg a \to \neg b$, which translates to "If I do not study hard, then I will not pass the course."
- The converse of $a \to b$ is $b \to a$, which translates to "If I pass the course, then I studied hard."
- The contrapositive of $a \to b$ is $\neg b \to \neg a$, which translates to "If I do not pass the course, then I did not study hard."

5. Classify the following expressions as tautologies, contradictions or contingencies using truth tables.

   (a) $((a \wedge \neg b) \wedge (\neg a \wedge c)) \to (b \wedge \neg c)$

   Solution: First we construct a truth table.

a	b	c	$a \wedge \neg b$	$\neg a \wedge c$	$(a \wedge \neg b) \wedge (\neg a \wedge c)$	$b \wedge \neg c$	$((a \wedge \neg b) \wedge (\neg a \wedge c)) \to (b \wedge \neg c)$
T	T	T	F	F	F	F	T
T	T	F	F	F	F	T	T
T	F	T	T	F	F	F	T
T	F	F	T	F	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	T	T
F	F	T	F	T	F	F	T
F	F	F	F	F	F	F	T

   Since every entry in the last column is true, the statement is a tautology.

   (b) $\neg((a \to \neg b) \to (b \to \neg a))$

   Solution: First we construct a truth table.

a	b	$a \to \neg b$	$b \to \neg a$	$(a \to \neg b) \to (b \to \neg a)$	$\neg((a \to \neg b) \to (b \to \neg a))$
T	T	F	F	T	F
T	F	T	T	T	F
F	T	T	T	T	F
F	F	T	T	T	F

   Since every entry in the last column is false, the statement is a contradiction.

6. Classify the following expressions as tautologies, contradictions or contingencies using logical equivalences.

   (a) $(p \wedge q) \to (p \to q)$

   Solution: Apply the following logical equivalences.

$$
\begin{array}{lll}
 & (p \wedge q) \to (p \to q) & \\
\equiv & \neg(p \wedge q) \vee (\neg p \vee q) & \text{Implication Equivalence} \\
\equiv & (\neg p \vee \neg q) \vee (\neg p \vee q) & \text{De Morgan's Law} \\
\equiv & (\neg p \vee \neg p) \vee (\neg q \vee q) & \text{Commutative, Associative Laws} \\
\equiv & \mathrm{T} \vee (\neg q \vee q) & \text{Negation Law} \\
\equiv & \mathrm{T} & \text{Domination Law}
\end{array}
$$

   Since the expression is logically equivalent to true, it is a tautology.

   (b) $\neg((\neg a \to (b \to c)) \to (b \to (a \vee c)))$

   Solution: Apply the following logical equivalences.

$$
\begin{array}{lll}
 & \neg((\neg a \to (b \to c)) \to (b \to (a \vee c))) & \\
\equiv & \neg((\neg\neg a \vee (b \to c)) \to (\neg b \vee (a \vee c))) & \text{Implication Equivalence (twice)} \\
\equiv & \neg((a \vee (b \to c)) \to (\neg b \vee (a \vee c))) & \text{Double Negation} \\
\equiv & \neg((a \vee (\neg b \vee c)) \to (\neg b \vee (a \vee c))) & \text{Implication Equivalence} \\
\equiv & \neg(\neg(a \vee (\neg b \vee c)) \vee (\neg b \vee (a \vee c))) & \text{Implication Equivalence} \\
\equiv & \neg(\neg(a \vee (\neg b \vee c)) \vee (a \vee (\neg b \vee c))) & \text{Commuative, Associative Laws} \\
\equiv & \neg \mathrm{T} & \text{Negation Law} \\
\equiv & \mathrm{F} & \text{by definition}
\end{array}
$$

Since the expression is logically equivalent to false, it is a contradiction.

7. Determine if the following pairs of expressions are logically equivalent. You may *not* use truth tables.

   (a) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

   Solution: Apply the following logical equivalences, starting with $(p \wedge q) \vee (\neg p \wedge \neg q)$.

$$
\begin{array}{lll}
 & (p \wedge q) \vee (\neg p \wedge \neg q) & \\
\equiv & ((p \wedge q) \vee \neg p) \wedge ((p \wedge q) \vee \neg q)) & \text{Distributive Law} \\
\equiv & ((p \vee \neg p) \wedge (\neg p \vee q)) \wedge ((\neg q \vee p) \wedge (q \vee \neg q)) & \text{Distributive Law (twice)} \\
\equiv & (\mathrm{T} \wedge (\neg p \vee q)) \wedge ((\neg q \vee p) \wedge \mathrm{T}) & \text{Negation Law (twice)} \\
\equiv & (\neg p \vee q) \wedge (\neg q \vee p) & \text{Identity Law (twice)} \\
\equiv & (p \to q) \wedge (q \to p) & \text{Implication Equivalence} \\
\equiv & p \leftrightarrow q & \text{Biconditional Equivalence}
\end{array}
$$

   This series of logical equivalences shows that the first statement is logically equivalent to the last. Therefore, $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

   (b) $\neg p \to (q \to r)$ and $q \to (p \vee r)$

   Solution: Apply the following logical equivalences, starting with $\neg p \to (q \to r)$.

$$
\begin{array}{lll}
 & \neg p \to (q \to r) & \\
\equiv & \neg\neg p \vee (q \to r) & \text{Implication Equivalence} \\
\equiv & p \vee (q \to r) & \text{Double Negation} \\
\equiv & p \vee (\neg q \vee r) & \text{Implication Equivalence} \\
\equiv & \neg q \vee (p \vee r) & \text{Associative, Commutative Laws} \\
\equiv & q \to (p \vee r) & \text{Implication Equivalence}
\end{array}
$$

   This series of logical equivalences shows that the first statement is logically equivalent to the last. Therefore, $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

8. Translate the following logical expressions into English, where $L(x)$ is "x is a lion" and $M(x)$ is "x eats meat."

   (a) $\forall x(L(x) \to M(x))$

   Solution: The universe of discourse is all things. The statement says "All lions eat meat."

   (b) $\forall x(L(x) \wedge M(x))$

   Solution: The universe of discourse is all things. The statement says "Every living thing is a lion who eats meat."

   (c) $\neg\forall x(\neg M(x) \to \neg L(x))$

   Solution: In this case, it helps to simplify the statement a little bit first. Apply quantifier negation to get $\exists x \neg(\neg M(x) \to \neg L(x))$. Apply Implication Equivalence to get $\exists x \neg(\neg\neg M(x) \vee \neg L(x))$. Now apply De Morgan's Law to get $\exists x(\neg\neg\neg M(x) \wedge \neg\neg L(x))$. Finally, apply double negation to get $\exists x(\neg M(x) \wedge L(x))$. The statement says "There is at least one lion who does not eat meat."

9. Translate the following English statements into logical expressions using predicates. The universe of discourse is the set of all people.

(a) Every mathematician is good at calculus but not every mathematician is good at singing.

Solution: Let $M(x)$ denote "x is a mathematician," $C(x)$ denote "x is good at calculus" and $S(x)$ denote "x can sing." The statement is $\forall x(M(x) \to C(x)) \land \exists x(M(x) \land \neg S(x))$. This could also be written $\forall x(M(x) \to C(x)) \land \neg\forall x(M(x) \to S(x))$

(b) Not everyone who is a good musician is rich.

Solution: Let $M(x)$ denote "x is a good musician" and $R(x)$ denote "x is rich." The statement is $\neg\forall x(M(x) \to R(x))$. This could also be written as $\exists x(M(x) \land \neg R(x))$.

(c) There are exactly two people in this class who are good at calculus and are good at discrete mathematics.

Solution: The easiest way to translate this is to break it down into two components. First, state that there are *at least* two people in this class who are good at calculus and are good at discrete mathematics, and then state that there are *at most* two (*i.e.*, not three) people in this class who are good at calculus and are good at discrete mathematics. If both of these are true simultaneously, then the situation is the same as in the original statement. Let $C(x)$ denote "x is in this class," $L(x)$ denote "x is good at calculus" and $D(x)$ denote "x is good at discrete mathematics."

To say that *at least* two people in this class who are good at calculus and are good at discrete mathematics, we write $\exists x\exists y(C(x) \land C(y) \land L(x) \land L(y) \land D(x) \land D(y) \land (x \neq y))$.

To say that there are not three distinct people in this class who are good at calculus and are good at discrete mathematics, we write $\neg\exists x\exists y\exists z C(x) \land C(y) \land C(z) \land L(x) \land L(y) \land L(z) \land D(x) \land D(y) \land D(z) \land (x \neq y) \land (x \neq z) \land (y \neq z))$.

Putting these together, we have $(\exists x\exists y(C(x) \land C(y) \land L(x) \land L(y) \land D(x) \land D(y) \land (x \neq y))) \land (\neg\exists x\exists y\exists z C(x) \land C(y) \land C(z) \land L(x) \land L(y) \land L(z) \land D(x) \land D(y) \land D(z) \land (x \neq y) \land (x \neq z) \land (y \neq z)))$.

10. Determine which of the following arguments are valid and which ones are invalid. Justify your answer.

(a) All lions like to eat meat. My dog is not a lion. Therefore, my dog does not like to eat meat.

Solution: Let $L(x)$ denote "x is a lion" and $M(x)$ denote "x likes to eat meat." Let a denote my dog. The premises of the argument are $\forall x(L(x) \to M(x))$ and $\neg L(a)$ and the conclusion is $\neg M(a)$.

This argument is not valid. In order to show this, we need to set truth values such that all of the premises are true, but the conclusion is false. Let $L(x)$ be false for all elements x in the universe of discourse and let $M(x)$ be true for all elements x in the universe of discourse. Then the first premise is always true since the hypothesis of the implication inside the quantifier is always false, and so the implication is always true. Similarly, the second premise is true because $\neg L(a) \equiv \neg \mathrm{F} \equiv \mathrm{T}$. The conclusion, however, is false because $\neg M(a) \equiv \neg \mathrm{T} \equiv \mathrm{F}$. Since we found truth settings that make the premises true and the conclusion false, we know that the argument is invalid.

(b) If n is an integer and $n^2 > 20$ then $n > 4$.

Solution: The premises of the argument are "n is an integer" and "$n^2 > 20$." The conclusion is $n > 4$.

The argument is not valid. In order to show this, we need to set truth values such that all of the premises are true, but the conclusion is false. Take $n = -5$. Then the first premise is true because n is an integer. The second premise is true because $n^2 = (-5)^2 = 25 > 20$. However, the conclusion is not true because $n = -5 \leq 4$.

(c) Anyone who owns a computer knows how to use a word processor. Since Bob owns a computer, he knows how to use a word processor.

Solution: Let $C(x)$ denote "x owns a computer" and $W(x)$ denote "x knows how to use a word processor." Denote Bob by b. The premises of the argument are $\forall x(C(x) \rightarrow W(x))$ and $C(b)$. The conclusion is $W(b)$.

1.	$\forall x(C(x) \rightarrow W(x))$	
2.	$C(b)$	$\therefore W(b)$
3.	$C(b) \rightarrow W(b)$	Universal Instantiation of 1
4.	$W(b)$	Modus Ponens from 2,3

Since we were able to derive the conclusion of the argument from the premises using the rules of inference, the argument is valid.

11. Express the negation of the following statements in English. The universe of discourse is the set of all people.

(a) Everyone who sings can dance.

Solution: Let $S(x)$ denote "x can sing" and $D(x)$ denote "x can dance." The original statement is $\forall x(S(x) \rightarrow D(x))$. The negation is thus $\neg\forall x(S(x) \rightarrow D(x))$. By applying quantifier negation, we get $\exists x\neg(S(x) \rightarrow D(x))$. By applying Implication Equivalence, we get $\exists x\neg(\neg S(x) \vee D(x))$. By applying De Morgan's Law, we get $\exists x(S(x) \wedge \neg D(x))$. The negation of the original statement is "There exists a person who can sing but not dance."

(b) Some people do not know calculus.

Solution: Let $C(x)$ denote "x knows calculus." The original statement is $\exists x\neg C(x)$. The negation is thus $\neg\exists x\neg C(x)$. Applying quantifier negation, we get $\forall x\neg\neg C(x)$. Applying double negation, we get $\forall x C(x)$. The negation of the original statement is "Everyone knows calculus."