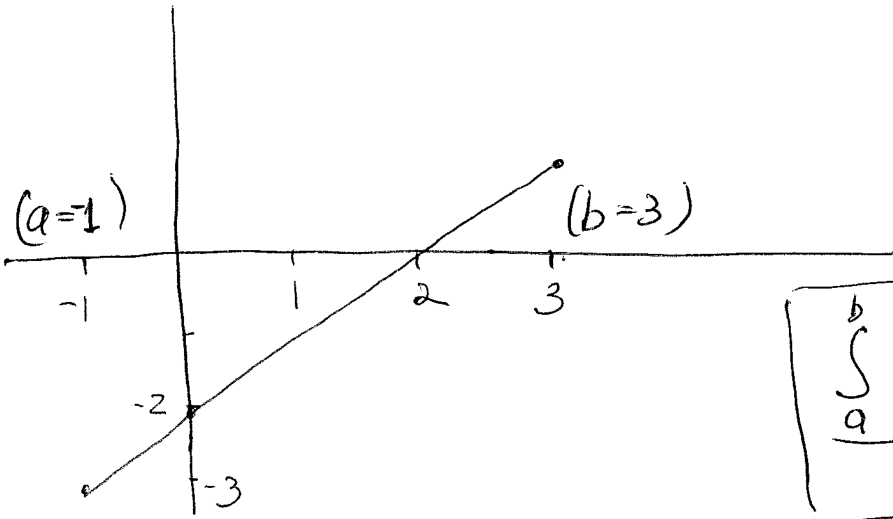
Example: $f(x) = x-2 \quad , \; x \in [-1, 3] = [a,b]$

Find $\bar{f}$.



$$\boxed{\frac{\int_a^b f(x)\,dx}{b-a} = \bar{f}}$$

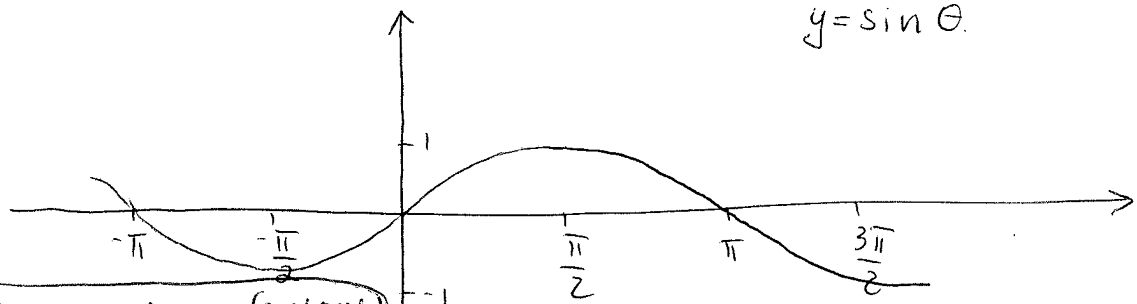
$$\bar{f} = \frac{\int_{-1}^{3} (x-2)\,dx}{3-(-1)} = \frac{\frac{x^2}{2}\Big|_{-1}^{3} - 2x\Big|_{-1}^{3}}{4} =$$

$$= \frac{\frac{9}{2} - \frac{(-1)^2}{2} - 2\big(3-(-1)\big)}{4} = \frac{4-2\cdot 4}{4} = -1$$

$$\left(\int_{-1}^{3} f(x)\,dx = \int_{-1}^{3} (x-2)\,dx = -4 \right)$$

$$\int_{-1}^{3} \bar{f}\,dx = \int_{-1}^{3} (-1)\,dx =$$

$$\int_{3}^{-1} dx = x \Big|_{3}^{-1} = (-1) - 3 = -4 = \bar{f}\cdot(b-a) = -1\cdot 4 = -4$$



a function: (input) $\longrightarrow$ (output)

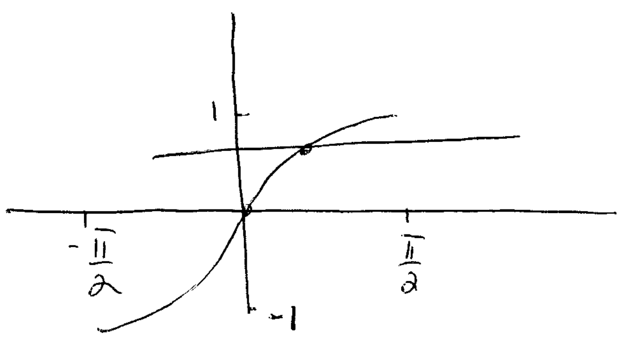
sin : $\theta \rightarrow y$

↑ an angle ↖ a number

$y = \sin\theta$, the domain is $\mathbb{R}$: $-\infty < \theta < \infty$, $\theta \in \mathbb{R}$

the range is $[-1, 1]$, $y \in [-1, 1]$

However, $y = \sin\theta$ is not one-to-one, because the horizontal line test is not satisfied. Let's restrict the domain. Consider $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



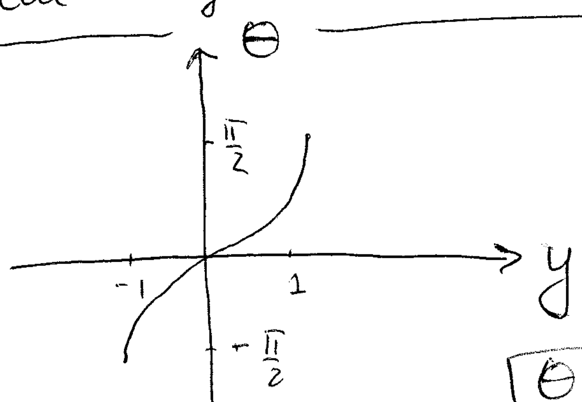
Now, the horizontal test is satisfied and the restricted function is invertible.

$\sin^{-1}$ or arcsin : $y \longrightarrow \theta$

a number an angle

$\theta = \arcsin y$, the domain is $[-1, 1]$

the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



(We reflected the graph of $\sin\theta$ about the diagonal)

$\boxed{\theta = \arcsin y}$

Goal:

$(\arcsin y)' - ?$

$\sin(\arcsin y) = y$ ← differentiate both sides with respect to y.

(On the LHS, we use the chain rule)

$$\left(\sin(\arcsin y)\right)' = 1$$

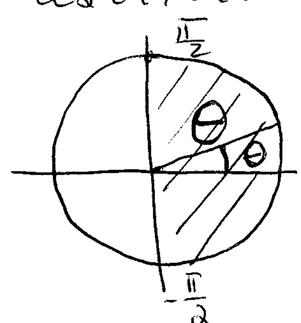
$$\cos(\arcsin y)\cdot(\arcsin y)' = 1$$

$$(\arcsin y)' = \frac{1}{\cos(\underbrace{\arcsin y}_{\theta})} = \frac{1}{\cos(\theta)}$$

Remember:
$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

Since we have the restriction on θ:

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\cos\theta \geq 0.$

Thus, we exclude $\cos\theta = -\sqrt{1-\sin^2\theta}$ (which is a negative number)

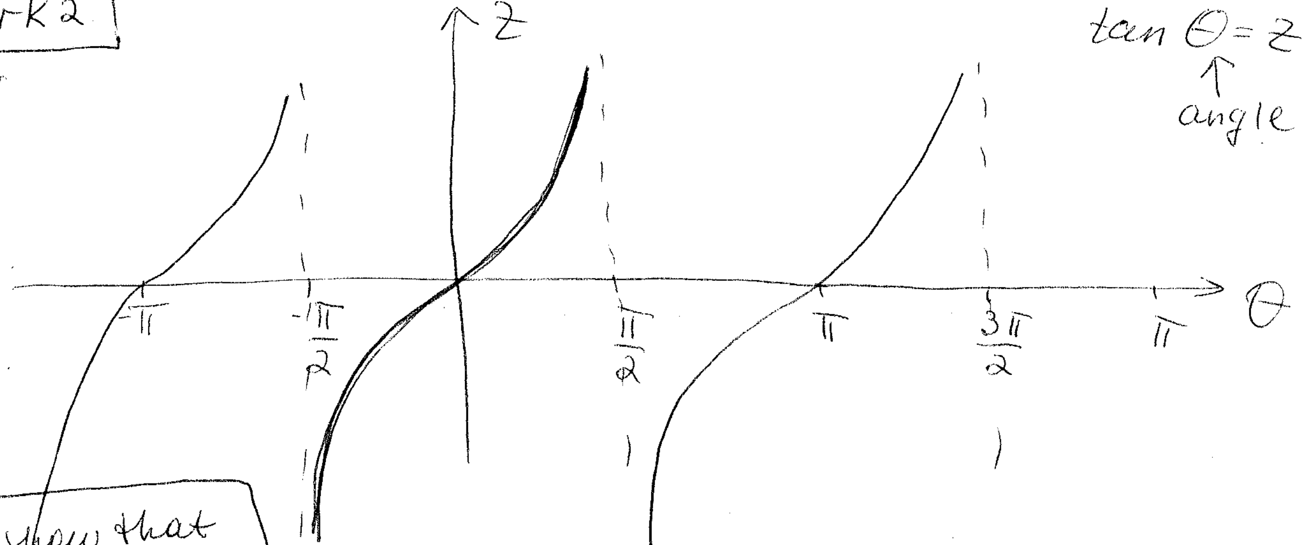
$\cos\theta = \sqrt{1-\sin^2\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Going back to $(\arcsin y)' = \dfrac{1}{\cos\theta} = \dfrac{1}{\sqrt{1-\sin^2\theta}} =$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} = \frac{1}{\sqrt{1-(\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1-y^2}}$$

$$\boxed{(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}} \Rightarrow \boxed{\int \frac{dy}{\sqrt{1-y^2}} = \arcsin(y) + C}$$

Remark 2



$\tan\theta = z$

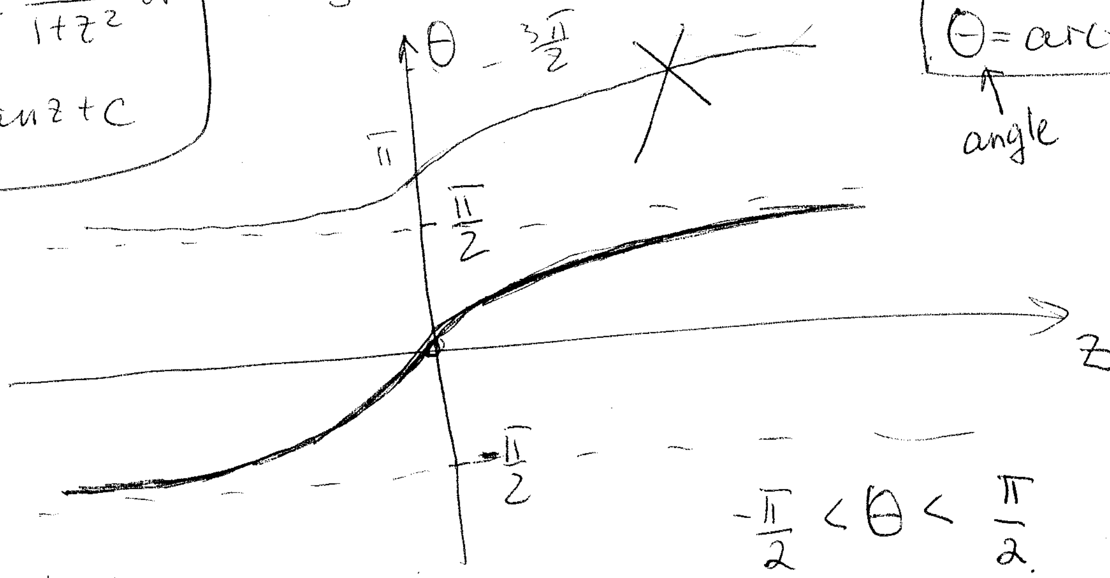
angle   number

Want to show that

$(\arctan z)' = \frac{1}{1+z^2}$ or

$\int \frac{dz}{1+z^2} = \arctan z + C$

reflect curves about $z=\theta$:



$\boxed{\theta = \arctan z}$

angle   number

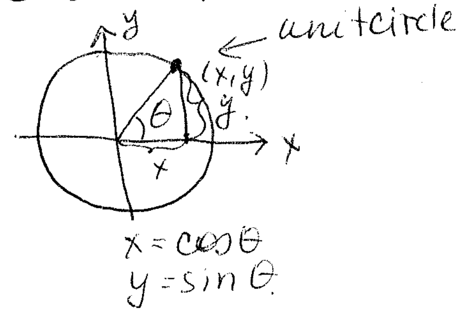
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Preliminaries:

(a) We use the following equality $\sin^2\theta + \cos^2\theta = 1$

multiply by $\frac{1}{\cos^2\theta}$:

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta} \quad \text{or} \quad \cos^2\theta = \frac{1}{1+\tan^2\theta}. \qquad (A)$$



$x = \cos\theta$
$y = \sin\theta$

(b) $(\tan\theta)' = \left(\frac{\sin\theta}{\cos\theta}\right)' = \frac{1}{\cos^2\theta}$.

To calculate $(\arctan z)'$, we differentiate $\tan(\arctan z) = z$ with respect to z:

$$\frac{1}{\cos^2(\arctan z)} \cdot (\arctan z)' = 1 \quad \text{or}$$

$$(\arctan z)' = \cos^2(\arctan z) \overset{(A)}{=} \frac{1}{1+\tan^2(\arctan z)} =$$

$$= \frac{1}{1+(\tan(\arctan z))^2} = \frac{1}{1+z^2}. \Rightarrow \boxed{\int \frac{dz}{1+z^2} = \arctan z + C}$$

Evaluate $\int_0^1 \arctan x \, dx$

Use integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\boxed{\begin{array}{ll} u = \arctan x & dx = dv \\ \frac{du}{dx} = \frac{1}{1+x^2} & x = v \\ du = \frac{dx}{1+x^2} & \end{array}}$$

$$\int_0^1 \arctan x \, dx = x \cdot \arctan x \Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2} =$$

$$= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_0^1 \frac{x\,dx}{1+x^2}$$

→ For $\int_0^1 \frac{x\,dx}{1+x^2}$ = use substitution

$$\begin{array}{l} 1+x^2 = y(x) \\ 2x\,dx = dy \\ x\,dx = \frac{dy}{2} \end{array} \qquad \begin{array}{c|c|c} x & 0 & 1 \\ \hline y & 1 & 2 \end{array}$$

$$= \int_1^2 \frac{dy}{2 \cdot y} = \frac{1}{2}\int_1^2 \frac{dy}{y} = \frac{1}{2} \ln|y| \Big|_1^2 = \frac{1}{2}\left(\ln 2 - \ln 1\right) = \frac{1}{2}\ln 2.$$

back to the original integral:

$$\int_0^1 \arctan x \, dx = 1 \cdot \frac{\pi}{4} - 0 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.439$$

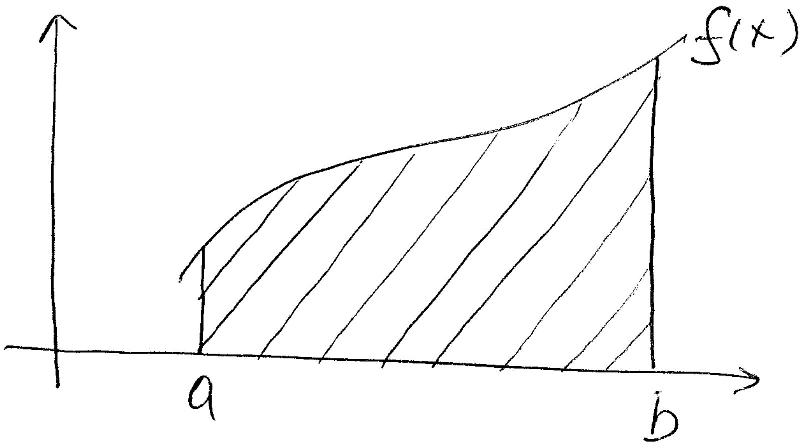
$$\int_0^1 \frac{e^{\arctan x}}{1+x^2} = \text{substitution}$$

$$\begin{array}{l} \arctan x = u(x) \\ \frac{dx}{1+x^2} = du \end{array} \qquad \begin{array}{c|c|c} x & 0 & 1 \\ \hline u & 0 & \frac{\pi}{4} \end{array}$$

$$= \int_0^{\frac{\pi}{4}} e^u\,du = e^u \Big|_0^{\frac{\pi}{4}} = e^{\frac{\pi}{4}} - e^0 = e^{\frac{\pi}{4}} - 1$$

Integrals and Areas

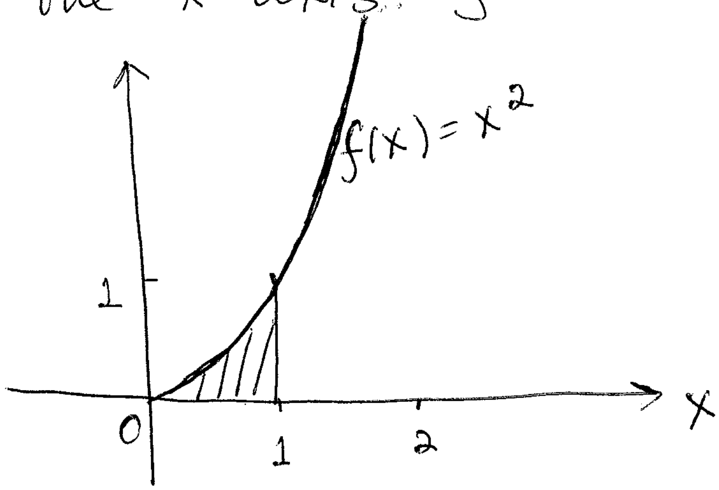
From Lecture 1, we know that if $f(x)$ is a continuous, nonnegative function ($f(x) \geq 0$) on $[a, b]$ then $\int_a^b f(x)dx$ = area between the x-axis and the graph of $f(x)$ between $x = a$ and $x = b$.



(follows from the Riemann sums)

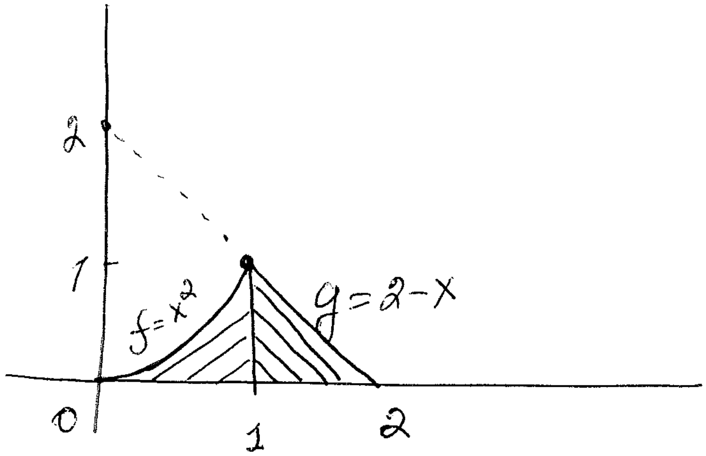
Example 1: $y = x^2 = f(x)$

Find the area between the graph of $f(x)$, the x-axis from $x = 0$ to $x = 1$.



$$\text{Area} = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

lets make the problem harder:
consider a piecewise, nonnegative function



This is a continuous function for $0 \le x \le 2$.

First, we need to find the point of intersection where the two curves indersect; we solve

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$D = 1 - 4(-2) = 9$$
$$x_1 = \frac{-1+3}{2} = 1$$
$$x_2 = \frac{-1-3}{2} = -2 \leftarrow$$

$x_2 = -2$ does not belong to the interval of integration $[0, 2]$

We consider the shaded region as a sum of the two regions: between $f(x)$ and the x-axis, $0 \le x \le 1$ and between $g(x)$, the x-axis, $1 \le x \le 2$.

$$\text{Area} = \text{Area } 1 + \text{Area } 2 = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx =$$

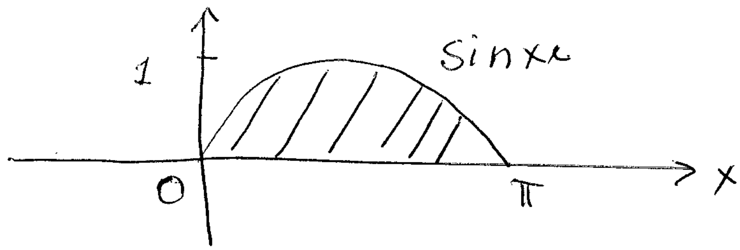
$$= \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{3} + \left(2\cdot 2 - \frac{2^2}{2}\right) - \left(2\cdot 1 - \frac{1^2}{2}\right) =$$

$$= \frac{1}{3} + (4-2) - (2-0{,}5) = \frac{1}{3} + 2 - \frac{3}{2} =$$

$$= \frac{2+12-9}{6} = \frac{5}{6}$$

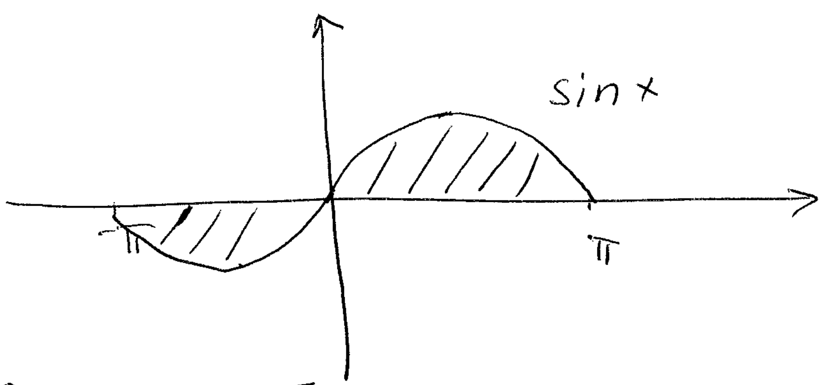
$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = -\cos\pi - (-\cos 0) =$$

$$-(-1) + 1 = 2$$



Consider $-\pi \le x \le \pi$

$$\int_{-\pi}^{\pi} \sin x\,dx = \int_{-\pi}^{0} \sin x\,dx + \int_0^{\pi} \sin x\,dx =$$



the previous example

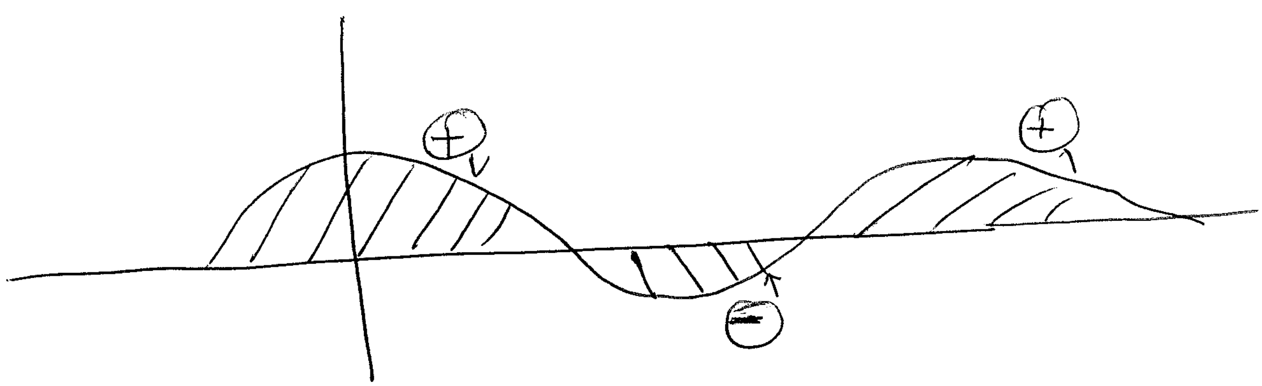
$$= -\cos x \Big|_{-\pi}^{0} + -\cos x \Big|_0^{\pi} = -\cos(0) + \cos(-\pi) + 2 =$$

$$= -1 - 1 + 2 = 0$$

How can an area be zero?

From Lecture 1, we know that when $f(x)$ is positive for some values of x and negative for others and $a < b$ then

$$\int_a^b f(x)\,dx = \begin{bmatrix}\text{sum of the areas of the regions} \\ \text{above the x-axis}\end{bmatrix} - \begin{bmatrix}\text{sum of the areas of the regions below} \\ \text{the x-axis}\end{bmatrix}$$



or

$$\int_a^b f(x)\,dx = \text{"area above"} - \text{"area below"}.$$

$$\int_{-\pi}^{\pi} \sin x\,dx = \underbrace{\int_{-\pi}^{0} \sin x\,dx}_{\text{negative area } = -2} + \underbrace{\int_{0}^{\pi} \sin x\,dx}_{\text{positive area } = 2} = 0.$$

↑ below the x-axis

↑ above the x-axis

Thus, <u>the definite integral</u> is the $\boxed{\text{signed area}}$

To find actual area between the graph of $f(x) = \sin x$, the x-axis and $x = -\pi$, $x = \pi$, we consider the following integral:

$$\int_{-\pi}^{\pi} |\sin x|\,dx = \int_{-\pi}^{0} |\sin x|\,dx + \int_{0}^{\pi} |\sin x|\,dx =$$

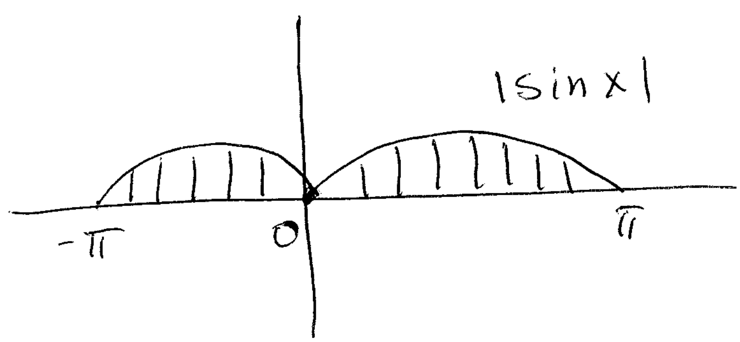
$$= \int_{-\pi}^{0} -\sin x\,dx + \int_{0}^{\pi} \sin x\,dx =$$

$$= -\int_{-\pi}^{0} \sin x\,dx + \int_{0}^{\pi} \sin x\,dx =$$

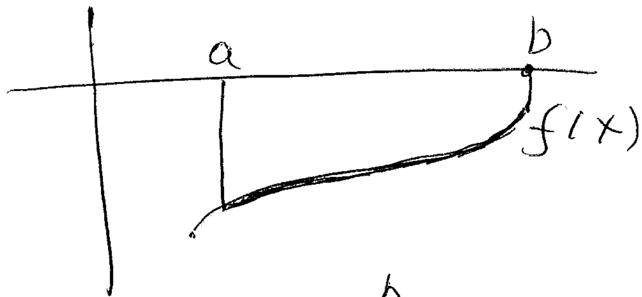
$$\cos x \Big|_{-\pi}^{0} - \cos x \Big|_{0}^{\pi} =$$

$$= \cos 0 - \cos(-\pi) - \cos \pi + \cos 0 =$$

$$= 1 + 1 + 1 + 1 = 4$$

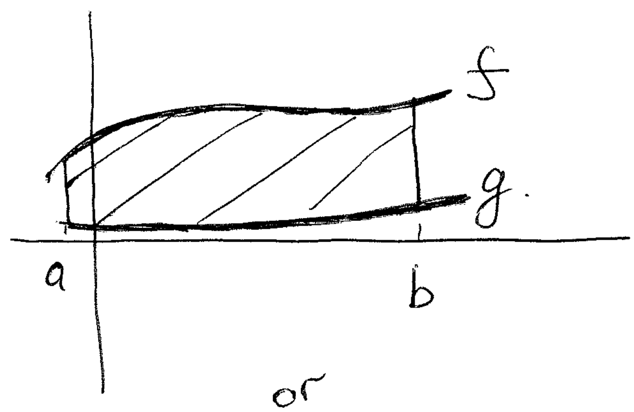


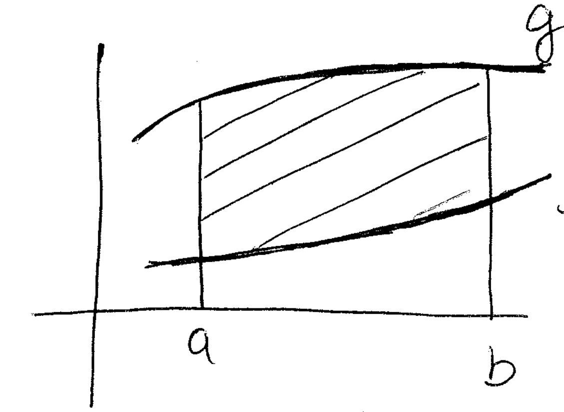
In the general case, if $f(x) \le 0$.



$$\text{Area} = \int_a^b |f(x)|\,dx = \int_a^b -f(x)\,dx = \underbrace{-\underbrace{\int_a^b f(x)\,dx}_{\text{negative}}}_{\text{positive}}$$

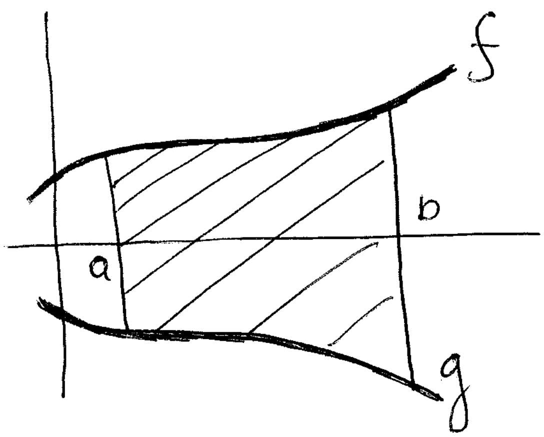
Let's make the problem harder and consider the area of the region between the graphs of continuous functions $f(x)$ and $g(x)$ bounded by the vertical lines $x=a$ and $x=b$.

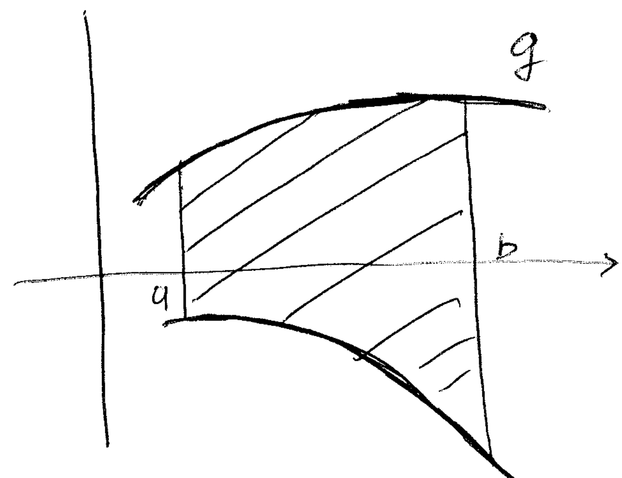




or

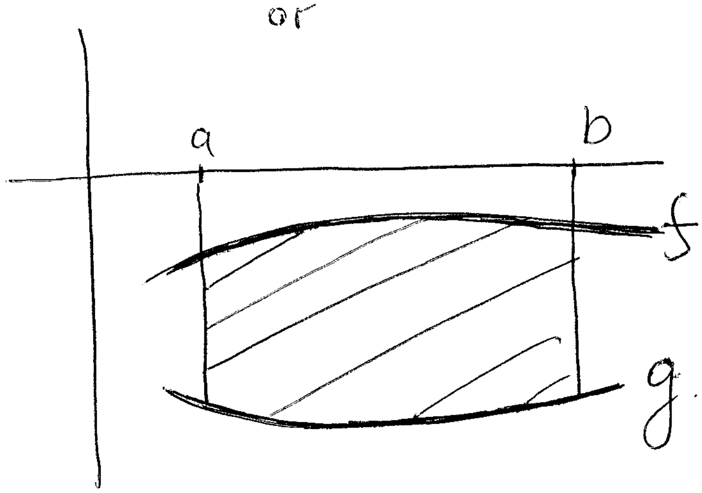
or



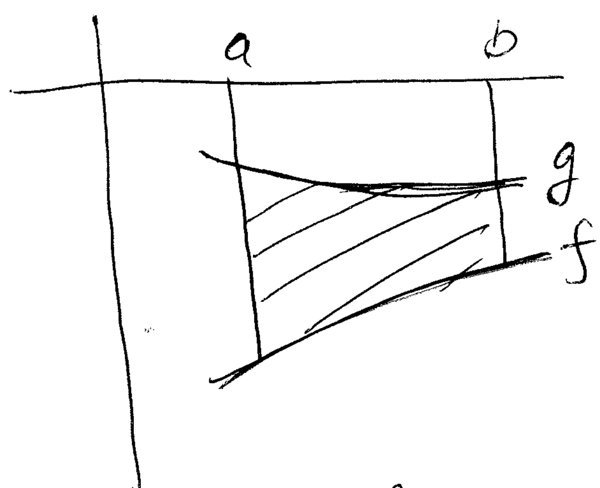


or

or



$f>g$



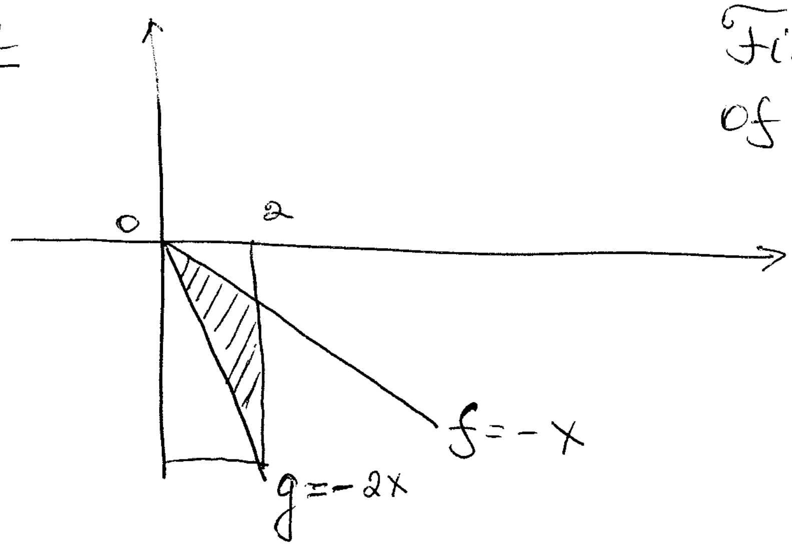
$g<f$

Then the area $= \int_a^b |f(x)-g(x)|\,dx$

(no point of intersection between f and f, $x \in [a,b)$)

$$\int_a^b | f(x)-g(x)|\,dx = \begin{cases} \int_a^b (f(x)-g(x))\,dx, & \text{if } f(x)>g(x) \text{ for } x\in[a,b] \\ \text{and} & \\ \int_a^b (g(x)-f(x))\,dx, & \text{if } g(x)>f(x) \text{ for } x\in[a,b] \end{cases}$$

Example



Find the area of the shaded region b/n $-x$, $-2x$, $0 \le x \le 2$

$\int_0^2 | f(x)-g(x)|\,dx =$ take a test point $0<\bar{c}<2$

$\bar{c} = 1$

$f(1)=-1$, $g(1)=-2$

$f(1) > g(1)$

and $f(x) > g(x)$ for $x \in (0,2)$,

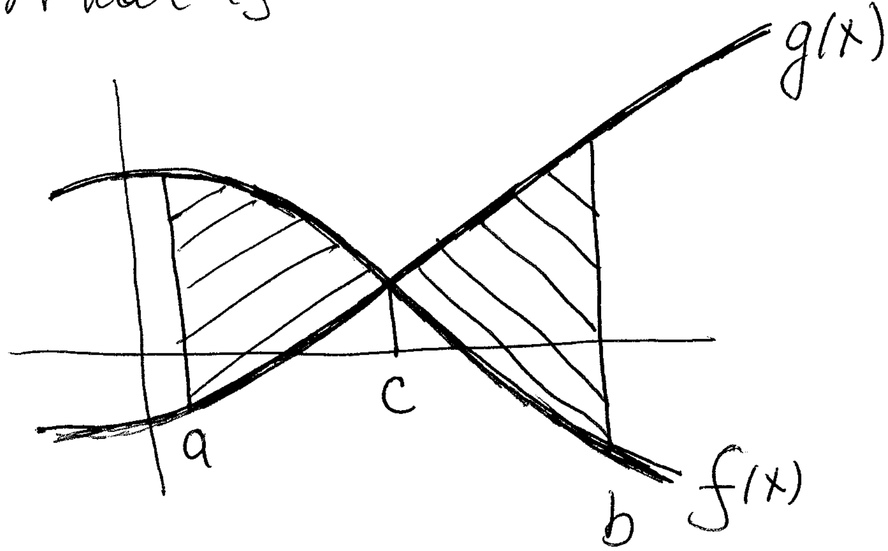
thus $|f(x) - g(x)| = f(x) - g(x)$.

$$\int_0^2 |f(x) - g(x)|\,dx = \int_0^2 (f(x) - g(x))\,dx =$$

$$= \int_0^2 \left(-x - (-2x)\right)dx = \int_0^2 (-x + 2x)\,dx = \int_0^2 x\,dx =$$

$$= \frac{x^2}{2}\Big|_0^2 = \frac{2^2}{2} - 0 = 2 \text{ (units}^2\text{)}.$$

→ What if the two functions intersect?



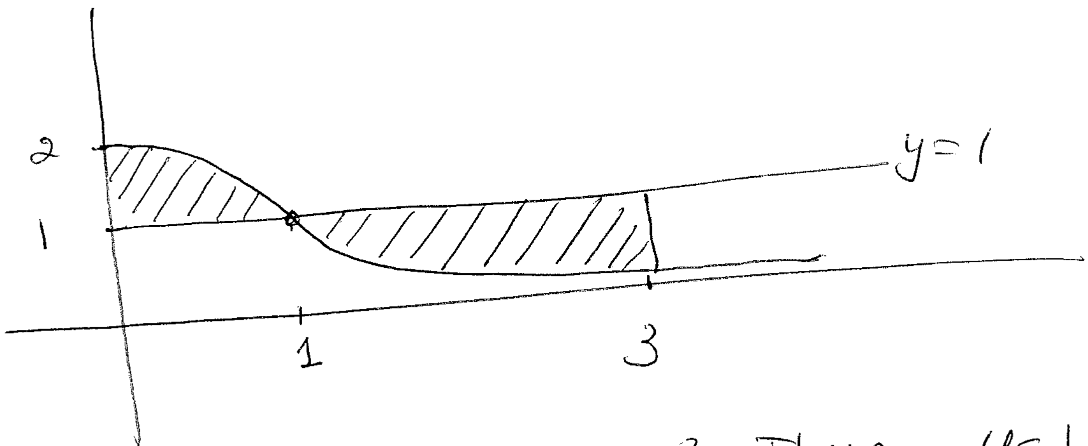
c is the point of intersection. (common point for both f(x) and g(x): $f(c) = g(c)$)

$$\int_a^b |f(x) - g(x)|\,dx = \int_a^c |f(x) - g(x)|\,dx + \int_c^b |f(x) - g(x)|\,dx =$$

$$= \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx.$$

$\boxed{\text{Example}}$ Find the area of the region between $y = \frac{2}{1+x^2} = f$ and

$y = 1 = g$ and $x=0, x=3$



$y(0)=2$

$y(x) = \frac{2}{1+x^2} \to 0$ as $x \to \infty$

Thus, $y=1$ intersects $y = \frac{2}{1+x^2}$

$y'(x) = -\frac{4x}{(1+x^2)^2} < 0$ for $x>0$. $\Rightarrow y = \frac{2}{1+x^2}$ is decreasing for $x>0$.

Points of intersection:

$\frac{2}{1+x^2} = 1$

$2 = 1+x^2$

$x^2 = 1 \Rightarrow x = \pm 1$ Within the interval $[0,3]$ the two curves intersect at $\underline{x=1 \text{ only}}$.

$$\int_0^3 |f(x)-g(x)|dx = \int_0^1 |f(x)-g(x)|dx + \int_1^3 |f(x)-g(x)|dx$$

In $[0,1]$ $\bar{c} = 0.5$ $f(0.5) = \frac{2}{1+0.25} = 1.6 > 1$

Thus, $f(x) = \frac{2}{1+x^2}$ lies above $g(x)=1$.

On $[1,3]$, take $\bar{c} = 1.5 \in [1,3)$

$$f(1.5) = \frac{2}{1+1.25} < 1$$

Thus, $f(x) = \frac{2}{1+x^2}$ lies below $g(x) = 1$ on $[1,3]$.

$$\text{Area} = \int_0^1 (f(x)-g(x))\,dx + \int_1^3 (g(x)-f(x))\,dx =$$

$$= \int_0^1 \left(\frac{2}{1+x^2} - 1\right)dx + \int_1^3 \left(1 - \frac{2}{1+x^2}\right)dx =$$

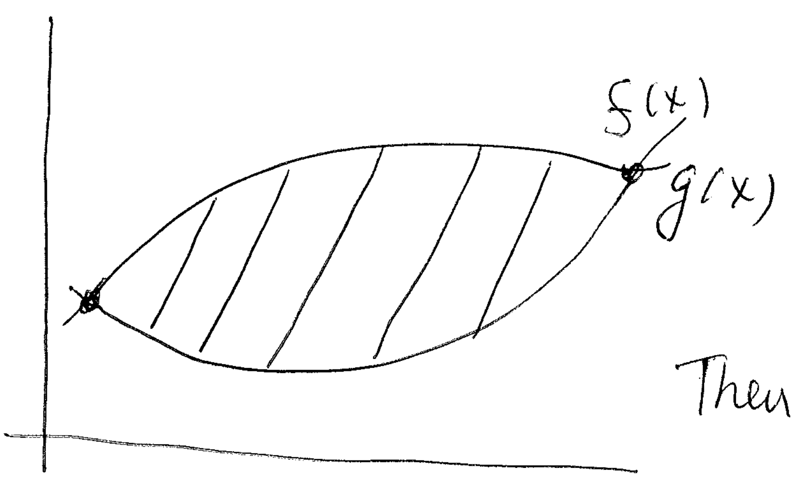
$$= 2\arctan x \Big|_0^1 - x\Big|_0^1 + x\Big|_1^3 - 2\arctan x\Big|_1^3 =$$

$$= 2\arctan 1 - 2\arctan 0 \underbrace{-1+0+3-1}\, - 2\arctan 3 +$$

$$+ 2\arctan 1 = 2\cdot\frac{\pi}{4} - 2\cdot 0 + 1 - 2\arctan 3 + 2\cdot\frac{\pi}{4} =$$

$$= \pi + 1 - 2\arctan 3 \approx 1.644$$

the last case:



Need to find the points of intersection (a, and b) (Usually they are not given).

Then $\text{Area} = \int_a^b |f(x)-g(x)|\,dx$

Integrals and Averages

The average value of a function f (denote by $\bar{f}$) on $[a, b]$ is

$$\bar{f} = \frac{\int_a^b f(x)\,dx}{b-a}$$

If $f \geqslant 0$, then the area under $\bar{f}$ = the area under f

$$\bar{f} \cdot (b-a) = \int_a^b f(x)\,dx$$