

The FTC

If f is continuous on $[a, b]$ and

$f(t) = F'(t)$ then

$$\int_a^b \frac{dF}{dt} dt = \int_a^b f(t) dt = \int_a^b f(t) dt \Big|_{t=a}^{t=b} =$$

family of
antiderivatives of $f(t)$

$$= (F(t) + C) \Big|_{t=a}^{t=b} = (F(b) + C) - (F(a) + C) =$$

$$= F(b) - F(a)$$

The definite integral of a rate of change gives the total change (or a shorthand notation,

$$\int_a^b f(t) dt = \underbrace{F(b) - F(a)}_{(F(t) + C) \Big|_{t=a}^{t=b}}$$

, where $F'(t) = f(t)$.

$$\int_2^3 t dt = \frac{t^2}{2} \Big|_{t=2}^{t=3} = \frac{t^2}{2} \Big|_2^3 =$$
$$= \frac{9}{2} - \frac{4}{2} = \frac{9-4}{2} = \frac{5}{2}$$

$$\int_5^7 \cos t dt = \sin t \Big|_5^7 = \sin(7) - \sin(5)$$

$$\int_5^7 \sin t dt = -\cos t \Big|_5^7 = -\cos(7) - (-\cos(5)) = \cos(5) - \cos(7)$$

Review of Integration Techniques

① A Collection of Indefinite Integrals we have to remember:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C, \quad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{dx}{\cos^2 x} = \tan(x) + C$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$

} (later)

② Guess - and - Check Method

The integrand is almost the same as the integrand from the collection (see part 1).
It is the same in sense that x is replaced by $ax+b$. For example

$$\rightarrow \int \frac{dx}{x-7} \text{ is almost like } \int \frac{dx}{x} = \ln|x| + C \rightarrow \begin{cases} \ln(x-7), x > 7 \\ \ln(7-x), x < 7 \end{cases}$$

Thus, we guess that $\int \frac{dx}{x-7} = \ln|x-7| + C$, then we check the answer: $(\ln|x-7| + C)' = \frac{1}{x-7}$.

or we can use the substitution:

$$x-7 = u(x)$$

$$dx = du$$

$$\int \frac{dx}{x-7} = \int \frac{du}{u} = \ln|u| + C = \ln|x-7| + C.$$

$$\rightarrow \int \cos(ax+b) dx \text{ is almost } \int \cos x dx = \sin x + C$$

We guess, that $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + C$.

$$\text{Check: } \left(\frac{\sin(ax+b)}{a} + C \right)' = \frac{\cos(ax+b) \cdot a}{a} = \cos(ax+b)$$

or use the substitution

$$ax+b = u(x)$$

$$adx = du \Rightarrow dx = \frac{du}{a}$$

$$\int \cos(ax+b) dx = \int \cos(u) \frac{du}{a} = \frac{1}{a} \int \cos(u) du = \frac{1}{a} \sin(u) + C = \frac{1}{a} \sin(ax+b) + C.$$

$$\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C$$

↑

To show that, we used guess-and-check method or the substitution $ax+b = u(x)$.

③ Integration by Parts

This technique is based on the product rule for derivatives. Integration by parts often succeeds with integrals like

$$\int x \sin x dx$$

$$\int x e^x dx$$

$$\int x^2 e^x dx$$

$\int \ln x dx$ which involve products.

$$\boxed{\int u dv = uv - \int v du}$$

Goal: To pick u, v in a such way that the RHS integral ($\int v du$) may be simpler than the LHS integral ($\int u dv$)

$$\boxed{\text{Example}} \quad \int \sqrt{x} \ln x dx$$

$$\int u dv = uv - \int v du$$

$$\sqrt{x} dx = dv = v' dx$$

$$\sqrt{x} = v'$$

$$v = \int \sqrt{x} dx = \frac{2x^{\frac{3}{2}}}{3}$$

$$u = \ln x$$

$$du = \frac{dx}{x}$$

$$\begin{aligned}
\int \sqrt{x} \ln x \, dx &= \frac{2}{3} x^{\frac{3}{2}} \ln x - \int \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{dx}{x} = \\
&= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} \, dx = \\
&= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \cdot x^{\frac{3}{2}} \cdot \frac{2}{3} + C = \\
&= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + C.
\end{aligned}$$

Integration of Definite Integrals

When we evaluate a definite integral, the FTC says that we must find an antiderivative of the integrand and then evaluate the antiderivative at the limits of integration.

$$\begin{aligned}
&F'(t) = f(t) \\
\int_a^b f(t) \, dt &= \int_a^b F'(t) \, dt \stackrel{\text{FTC}}{=} \int_a^b f(t) \, dt \Big|_{t=a}^{t=b} = \\
&= (F(t) + C) \Big|_{t=a}^{t=b} = (F(b) + C) - (F(a) + C) = \\
&= F(b) - F(a).
\end{aligned}$$

$$\int_0^4 2x \sqrt{x^2 + 1} dx = \text{substitution}$$

$$x^2 + 1 = u(x)$$

$$2x dx = du$$

x	0	4
u	1	17

$$= \int_1^{17} \sqrt{u} du = \int_1^{17} u^{\frac{1}{2}} du = \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} \Big|_{u=1}^{u=17}$$

$$= \frac{2}{3} (17)^{\frac{3}{2}} - \frac{2}{3} (1)^{\frac{3}{2}} = \frac{2}{3} \left[(17)^{\frac{3}{2}} - 1 \right]$$

$$\int_1^2 \frac{3x^2 + 1}{x^3 + x} dx =$$

the inner function is $(x^3 + x)$

$$x^3 + x = u(x)$$

$$3x^2 + 1 = \frac{du}{dx} \Rightarrow (3x^2 + 1) dx = du$$

x	1	2
u	2	10

$$= \int_2^{10} \frac{du}{u} = \ln|u| \Big|_{u=2}^{u=10} = \ln 10 - \ln 2 = \ln 5$$

$$\int_{\frac{1}{2}}^1 \frac{1}{x^2} e^{\frac{1}{x}} dx =$$

$$\frac{1}{x} = u(x)$$

$$-\frac{dx}{x^2} = du \Rightarrow \frac{dx}{x^2} = -du$$

x	$\frac{1}{2}$	1
u	2	1

$$= - \int_2^1 e^u du = \int_1^2 e^u du = e^u \Big|_{u=1}^{u=2} = e^2 - e$$

FTC

$$= \left(\int e^u du \Big|_{u=1}^{u=2} = (e^u + c) \Big|_{u=1}^{u=2} \right)$$

$$\int_0^{\frac{\pi}{6}} \cos x e^{\sin x} dx$$

$u = \sin x$		
$du = \cos x dx$		
x	0	$\frac{\pi}{6}$
u	0	$\frac{1}{2}$

$$= \int_0^{\frac{1}{2}} e^u du = e^u \Big|_{u=0}^{u=\frac{1}{2}} = e^{\frac{1}{2}} - 1.$$

$$\int_3^5 x \cos x dx =$$

$$\int u dv = uv - \int v du$$

$\cos x dx = dv = v' dx$
$\cos x = v'$
$v = \int \cos x dx = \sin x$
$x = y$
$dx = dy$

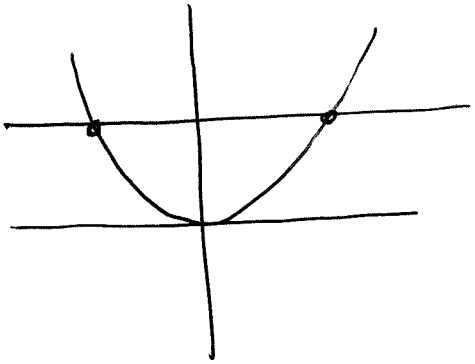
$$= x \sin x \Big|_{x=3}^{x=5} - \int_3^5 \sin x dy =$$

$$= 5 \sin 5 - 3 \sin 3 + \cos x \Big|_3^5 =$$

$$= 5 \sin 5 - 3 \sin 3 + \cos 5 - \cos 3$$

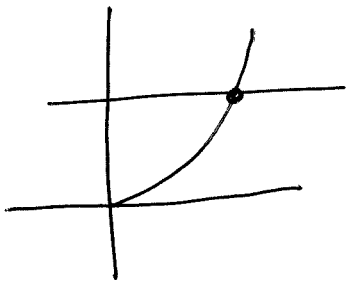
! (do not have to change the limits of integration when we use integration by parts technique)

$f(x) = x^2$ has no inverse; because it is not one-to-one function



Test The horizontal line test is not satisfied. Since a horizontal line intersects the graph of $f(x) = x^2$ more than one time.

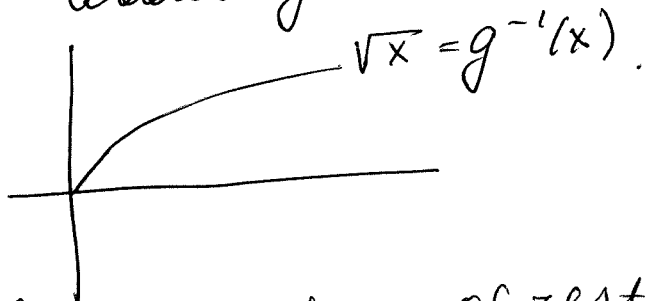
We can make the function invertible by restricting the domain of the function. $(-\infty, \infty)$
Consider only $x \geq 0$ (take a piece of $f(x)$)



← this function is one-to-one and thus it is invertible

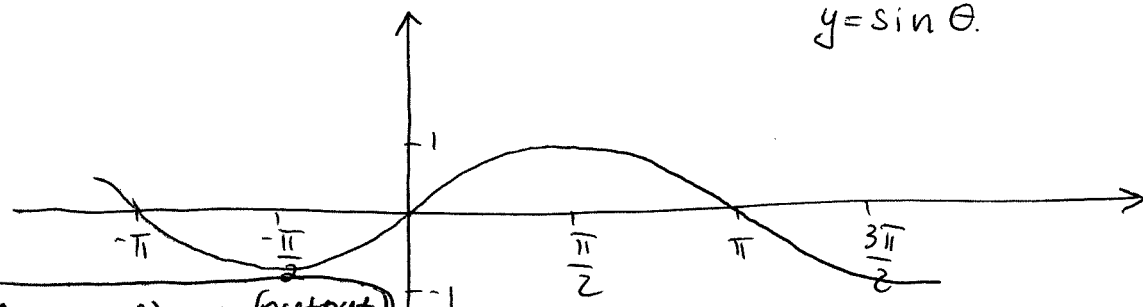
$$g(x) = x^2, x \geq 0$$

$g^{-1}(x) = \sqrt{x}$; we reflect about $y = x$:



→ The same idea of restriction of the domains of the trigonometric functions allows us to obtain the inverse trigonometric functions: $\arcsin x$, $\arctan x$ and so on.

$$y = \sin \theta$$

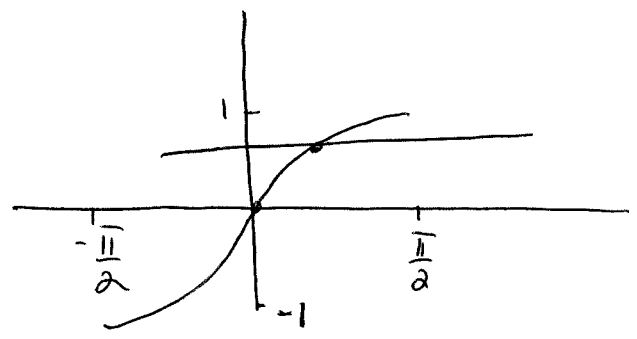


a function: (input) \rightarrow (output)

$$\sin : \begin{matrix} \theta & \rightarrow & y \\ \uparrow & & \uparrow \\ \text{an angle} & & \text{a number} \end{matrix}$$

$y = \sin \theta$, the domain is $\mathbb{R} : -\infty < \theta < \infty$
 $\theta \in \mathbb{R}$
 the range is $[-1, 1]$, $y \in [-1, 1]$

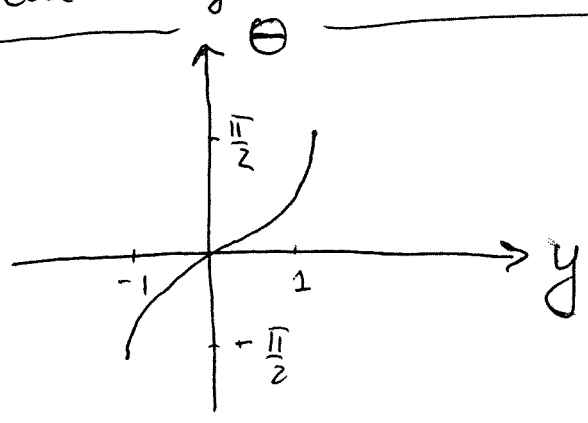
However, $y = \sin \theta$ is not one-to-one, because the horizontal line test is not satisfied.
 Let's restrict the domain. Consider $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



Now, the horizontal test is satisfied and the restricted function is invertible.

$$\sin^{-1} \text{ or arcsin} : \begin{matrix} y & \rightarrow & \theta \\ \uparrow & & \uparrow \\ \text{a number} & & \text{an angle} \end{matrix}$$

$\theta = \arcsin y$, the domain is $[-1, 1]$
 the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



(We reflected the graph of $\sin \theta$ about the diagonal,

Goal:

$$(\arcsin y)' = ?$$

$\sin(\arcsin y) = y$ ← differentiate both sides with respect to y .

(On the LHS, we use the chain rule)

$$(\sin(\arcsin y))' = 1$$

$$\cos(\arcsin y) \cdot (\arcsin y)' = 1$$

$$(\arcsin y)' = \frac{1}{\cos(\arcsin y)} = \frac{1}{\cos(\theta)}$$

Remember: $\cos^2 \theta + \sin^2 \theta = 1$

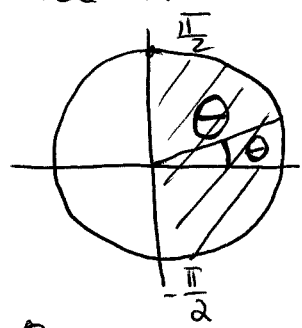
$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

Since we have the restriction on θ :

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos \theta \geq 0.$$



Thus, we exclude

$$\cos \theta = -\sqrt{1 - \sin^2 \theta} \text{ (which is a negative number)}$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta}, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Going back to } (\arcsin y)' = \frac{1}{\cos \theta} = \frac{1}{\sqrt{1 - \sin^2 \theta}} =$$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin y)}} = \frac{1}{\sqrt{1 - (\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1 - y^2}}$$

$$\boxed{(\arcsin y)' = \frac{1}{\sqrt{1 - y^2}}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{1 - y^2}} = \arcsin(y) + C$$