Carleton University * Faculty of Engineering and Design

Mechanics I

ECOR 1101

Lecture 21

EQUATIONS OF MOTION:
NORMAL AND TANGENTIAL COORDINATES
CYLINDRICAL COORDINATES

Sections 10.5-10.6

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EQUATIONS OF MOTION: NORMAL AND TANGENTIAL COORDINATES

Objectives:

Students will be able to:
1. Apply the equation of motion using normal and tangential coordinates.

APPLICATIONS

Race tracks are often banked in the turns to reduce the frictional forces required to keep the cars from sliding at high speeds.

If the car's maximum velocity and a minimum coefficient of friction between the tires and track are specified, how can we determine the minimum banking angle (θ) required to prevent the car from sliding?

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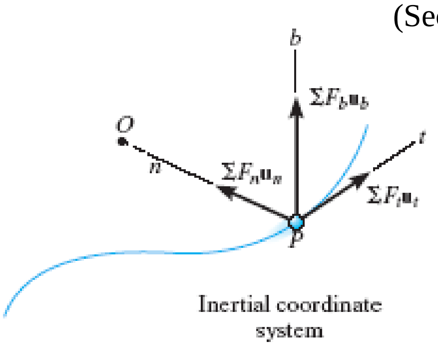
NORMAL & TANGENTIAL COORDINATES

(Section 13.5)



When a particle moves along a curved path, it is convenient to write the equation of motion in terms of normal and tangential coordinates.

The normal direction (n) always points toward the path's center of curvature.
In a circle, the center of curvature is the center of the circle.

The tangential direction (t) is tangent to the path, usually set as positive in the direction of motion of the particle.

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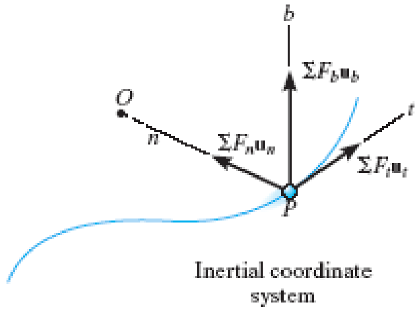
EQUATIONS OF MOTION



Since the equation of motion is a vector equation, $\sum \boldsymbol{F} = m\boldsymbol{a}$, it may be written in terms of the n & t coordinates as

$$\sum F_t \boldsymbol{u}_t + \sum F_n \boldsymbol{u}_n = m\boldsymbol{a}_t + m\boldsymbol{a}_n$$

Here $\sum F_t$ & $\sum F_n$ are the sums of the force components acting in the t & n directions, respectively.

This vector equation will be satisfied provided the individual components on each side of the equation are equal, resulting in the two scalar equations: $\sum F_t = ma_t$ and $\sum F_n = ma_n$.

Since there is *no motion in the binormal (b) direction*, we can also write $\sum F_b = 0$.

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NORMAL AND TANGENTIAL ACCERLERATIONS

The tangential acceleration, $a_t = dv/dt$, represents the time rate of change in the magnitude of the velocity.
Depending on the direction of $\sum F_t$, the particle's speed will either be increasing or decreasing.

The normal acceleration, $a_n = v^2/\rho$, represents the time rate of change in the direction of the velocity vector.

Remember, a_n always acts toward the path's center of curvature. Thus, $\sum F_n$ will always be directed toward the center of the path.

Recall, if the path of motion is defined as $y = f(x)$, the radius of curvature at any point can be obtained from

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

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SOLVING PROBLEMS WITH n-t COORDINATES

- Use n-t coordinates when a particle is moving along a curved path.
- Establish the n-t coordinate system on the particle.
- Draw free-body and kinetic diagrams of the particle.
  The normal acceleration (a_n) always acts "inward" (the positive n-direction).
  The tangential acceleration (a_t) may act in either the positive or negative t direction.
- Apply the equations of motion in scalar form and solve.
- It may be necessary to employ the kinematic relations:

$a_t = dv/dt = v\ dv/ds \qquad a_n = v^2/\rho$

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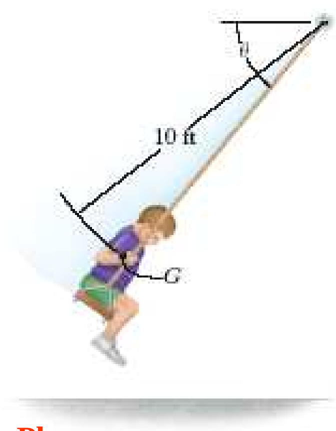

EXAMPLE

Given: At the instant θ = 60°, the boy's center of mass G is momentarily at rest.
The boy has a weight of 60 lb. Neglect his size and the mass of the seat and cords.

Find: The boy's speed and the tension in each of the two supporting cords of the swing when θ = 90°.

Plan:

1) Since the problem involves a curved path and finding the force perpendicular to the path, use n-t coordinates.
   Draw the boy's free-body and kinetic diagrams.
2) Apply the equation of motion in the n-t directions.
3) Use kinematics to relate the boy's acceleration to his speed.

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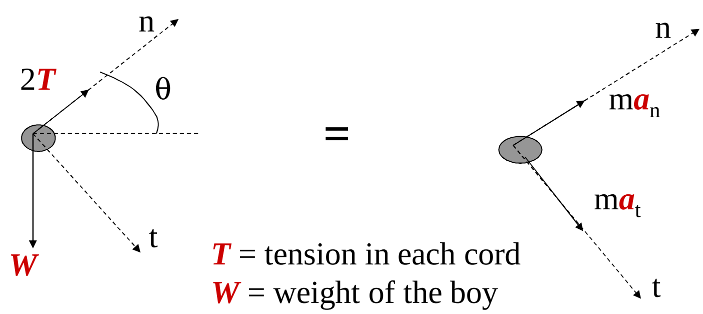
EXAMPLE
(continued)

Solution:

1) The n-t coordinate system can be established on the boy at some arbitrary angle θ.
Approximating the boy and seat together as a particle, the free-body and kinetic diagrams can be drawn.

Free-body diagram **Kinetic diagram**



T = tension in each cord
W = weight of the boy

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EXAMPLE
(continued)

2) Apply the equations of motion in the n-t directions.

(a) $\sum F_n = ma_n \Rightarrow 2T - W \sin\theta = ma_n$

Using $a_n = v^2/\rho = v^2/10$, w = 60 lb, and m = w/g = (60/32.2),

we get: $2T - 60 \sin\theta = (60/32.2)(v^2/10)$ (1)

(b) $\sum F_t = ma_t \Rightarrow W \cos\theta = ma_t$

$\Rightarrow 60 \cos\theta = (60/32.2)\ a_t$

Solving for a_t: $a_t = 32.2 \cos\theta$ (2)

Note that there are 2 equations and 3 unknowns (T, v, a_t). One more equation is needed.

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EXAMPLE
(continued)

3) Apply kinematics to relate a_t and v.

$v\,dv = a_t\,ds$ where $ds = \rho\,d\theta = 10\,d\theta$

$$\Rightarrow v\,dv = 32.2\cos\theta\,ds = 32.2\cos\theta\,(10\,d\theta)$$

$$\Rightarrow \int_0^v v\,dv = \int_{60}^{90} 322\cos\theta\,d\theta$$

$$\Rightarrow \frac{v^2}{2} = 322\sin\theta\,\Big|_{60}^{90} \Rightarrow v = 9.29 \text{ ft/s}$$

This v is the speed of the boy at $\theta = 90°$. This value can be substituted into equation (1) to solve for T.

$$2T - 60\sin(90°) = (60/32.2)(9.29)^2/10$$

T = 38.0 lb (the tension in each cord)

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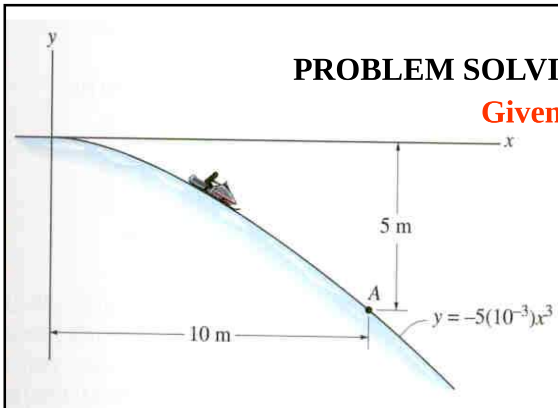
PROBLEM SOLVING



Given: A 200 kg snowmobile with rider is traveling down the hill. When it is at point A, it is traveling at 4 m/s and increasing its speed at 2 m/s^2.

Find: The resultant normal force and resultant frictional force exerted on the tracks at point A.

Plan:
1) Treat the snowmobile as a particle. Draw the free-body and kinetic diagrams.
2) Apply the equations of motion in the n-t directions.
3) Use calculus to determine the slope and radius of curvature of the path at point A.

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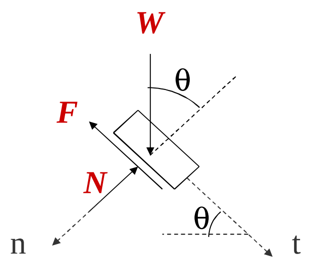
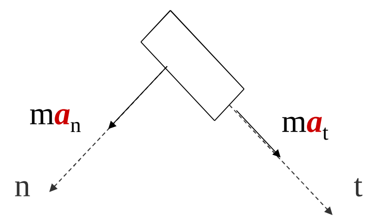
PROBLEM SOLVING (continued)

Solution:

1) The n-t coordinate system can be established on the snowmobile at point A. Treat the snowmobile and rider as a particle and draw the free-body and kinetic diagrams:





W = mg = weight of snowmobile and passenger
N = resultant normal force on tracks
F = resultant friction force on tracks

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PROBLEM SOLVING
(continued)

2) Apply the equations of motion in the n-t directions:

$\sum F_n = ma_n$ => $W \cos\theta - N = ma_n$

Using $W = mg$ and $a_n = v^2/\rho = (4)^2/\rho$

=> $(200)(9.81)\cos\theta - N = (200)(16/\rho)$

=> $N = 1962 \cos\theta - 3200/\rho$ (1)

$\sum F_t = ma_t$ => $W \sin\theta - F = ma_t$

Using $W = mg$ and $a_t = 2\ m/s^2$ (given)

=> $(200)(9.81)\sin\theta - F = (200)(2)$

=> $F = 1962 \sin\theta - 400$ (2)

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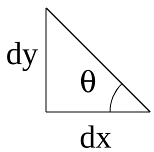
PROBLEM SOLVING
(continued)

3) Determine ρ by differentiating $y = f(x)$ at $x = 10$ m**:**

$y = -5(10^{-3})x^3 \Rightarrow dy/dx = (-15)(10^{-3})x^2 \Rightarrow d^2y/dx^2 = -30(10^{-3})x$

$$\rho\Big|_{x = 10\text{ m}} = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[1 + (-1.5)^2]^{3/2}}{|-0.3|} = 19.52\text{ m}$$

Determine θ from the slope of the curve at A:



$$\tan\theta = dy/dx\Big|_{x = 10\text{ m}}$$

$$\theta = \left|\tan^{-1}(dy/dx)\right| = \left|\tan^{-1}(-1.5)\right| = 56.31°$$

From Eq.(1): N = 1962 cos(56.31) – 3200/19.53 = 924 N

From Eq.(2): F = 1962 sin(56.31) – 400 = 1232 N

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**EQUATIONS OF MOTION:
CYLINDRICAL COORDINATES**

Objectives:

Students will be able to:

1. Analyze the kinetics of a particle using cylindrical coordinates.

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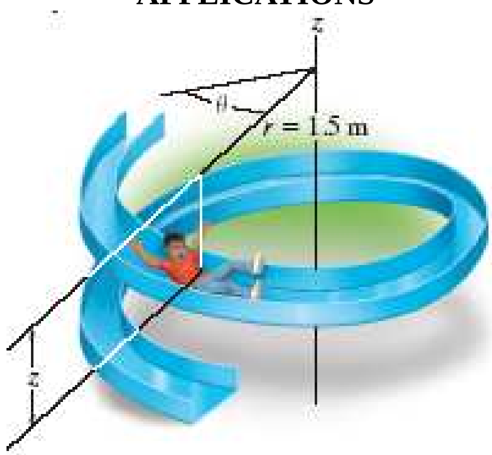
APPLICATIONS



The forces acting on the 100-lb boy can be analyzed using the cylindrical coordinate system.

<u>If the boy slides down at a constant speed of 2 m/s, can we find the frictional force acting on him?</u>

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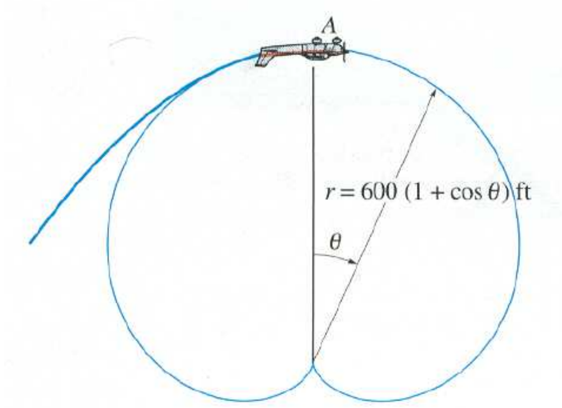
APPLICATIONS
(continued)



When an airplane executes the vertical loop shown above, the centrifugal force causes the normal force (apparent weight) on the pilot to be smaller than her actual weight.

<u>If the pilot experiences weightlessness at *A*, what is the airplane's velocity at *A*?</u>

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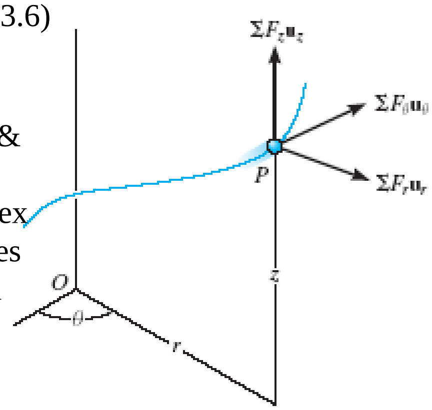
CYLINDRICAL COORDINATES
(Section 13.6)

This approach to solving problems has some external similarity to the normal & tangential method just studied. However, the path may be more complex or the problem may have other attributes that make it desirable to use cylindrical coordinates.



Equilibrium equations or "Equations of Motion" in cylindrical coordinates (using r, θ, and z coordinates) may be expressed in scalar form as:

$$\sum F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$\sum F_\theta = ma_\theta = m(r\ddot{\theta} - 2\dot{r}\dot{\theta})$$
$$\sum F_z = ma_z = m\ddot{z}$$

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CYLINDRICAL COORDINATES
(continued)

If the particle is constrained to move only in the r – θ plane (i.e., the z coordinate is constant), then only the first two equations are used (as shown below). The coordinate system in such a case becomes a polar coordinate system. In this case, the path is only a function of θ.

$$\sum F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$\sum F_\theta = ma_\theta = m(r\ddot{\theta} - 2\dot{r}\dot{\theta})$$

Note that a fixed coordinate system is used, not a "body-centered" system as used in the n – t approach.

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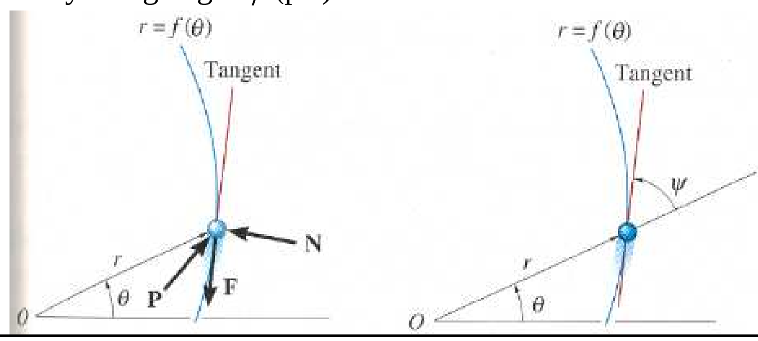
TANGENTIAL AND NORMAL FORCES

If a force ***P*** causes the particle to move along a path defined by $r = f(\theta)$, the normal force ***N*** <u>exerted by the path on the particle is always perpendicular to the path's tangent</u>.
The frictional force ***F*** <u>always acts along the tangent in the opposite direction of motion</u>.
The directions of ***N*** and ***F*** can be specified relative to the radial coordinate by using angle ψ (psi) .



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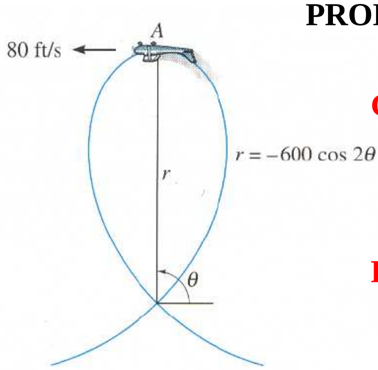
PROBLEM SOLVING



Given: A plane flies in a vertical loop as shown.
v_A = 80 ft/s (constant)
W = 130 lb

Find: Normal force on the pilot at A.

Plan: Determine $\dot{\theta}$ and $\ddot{\theta}$ from the velocity at A and by differentiating r. Solve for the accelerations, and apply the equation of motion to find the force.

Solution: Kinematics:

$$r = -600\cos(2\theta)$$
$$\dot{r} = 1200\sin(2\theta)\dot{\theta}$$
$$\ddot{r} = 2400\cos(2\theta)\dot{\theta}^2 + 1200\sin(2\theta)\ddot{\theta}$$

At A $(\theta = 90^\circ)$ $\quad \dot{r} = 0$

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PROBLEM SOLVING (continued)

Therefore $v_A = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2} = r\dot{\theta}$

Since $r = 600$ ft at A, $\dot{\theta} = \dfrac{80}{600} = 0.133\ \text{rad}/\text{s}$

Since v_A is constant, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \Rightarrow \ddot{\theta} = 0$

$$\ddot{r} = 2400\cos(180^\circ)\dot{\theta}^2 + 1200\sin(180^\circ)\ddot{\theta} = -2400(0.133)^2 = -42.67\ \text{ft}/\text{s}^2$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -42.67 - 600(0.133)^2 = -53.33\ \text{ft}/\text{s}^2$$

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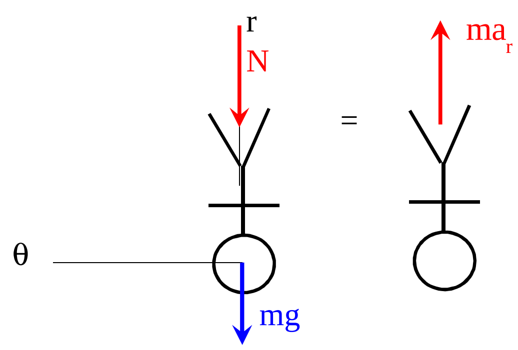
PROBLEM SOLVING (continued)

Free Body Diagram & Kinetic Diagram



Kinetics: $\sum F_r = ma_r \Rightarrow -mg - N = ma_r$

$$N = -130 - \frac{130}{32.2}(53.3) \Rightarrow N = 85.2\ \text{lb}$$

Notice that the pilot would experience weightlessness when his radial acceleration is equal to g.

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