Carleton University * Faculty of Engineering and Design

Mechanics I

ECOR 1101

Lecture 18

CURVILINEAR MOTION:
NORMAL AND TANGENTIAL COMPONENTS (n,t)
CYLINDRICAL COMPONENTS (r,θ)

Section 9.7-9.8

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CURVILINEAR MOTION:
NORMAL AND TANGENTIAL COMPONENTS

Objectives:

Students will be able to:

1. Determine the normal and tangential components of velocity and acceleration of a particle traveling along a curved path.

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APPLICATIONS

Cars traveling along a clover-leaf interchange experience an acceleration due to a change in speed, as well as due to a change in direction of the velocity.

If the car's speed is increasing at a known rate as it travels along a curve, how can we determine the magnitude and direction of its total acceleration?

Why would you care about the total acceleration of the car?

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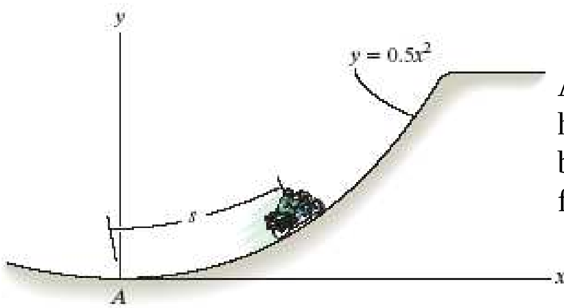
APPLICATIONS
(continued)



A motorcycle travels up a hill for which the path can be approximated by a function y = f(x).

If the motorcycle starts from rest and increases its speed at a constant rate, how can we determine its velocity and acceleration at the top of the hill?

How would you analyze the motorcycle's "flight" at the top of the hill?

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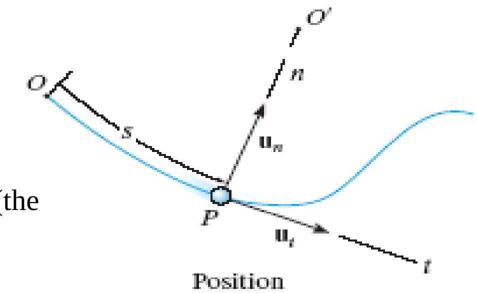
NORMAL AND TANGENTIAL COMPONENTS
(Section 12.7)

When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal (n) and tangential (t) coordinates are often used.



Position

In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle).

The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.
The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.

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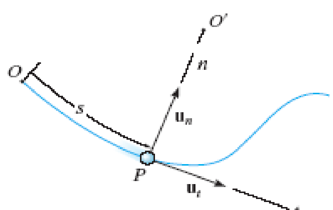
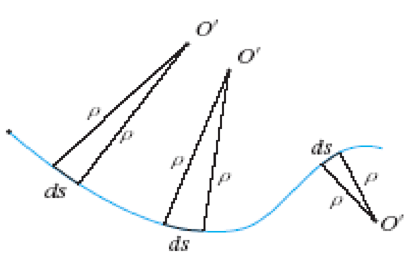
NORMAL AND TANGENTIAL COMPONENTS
(continued)



Position

The positive n and t directions are defined by the unit vectors $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$, respectively.



Radius of curvature

The center of curvature, O', always lies on the concave side of the curve. The radius of curvature, ρ, is defined as the perpendicular distance from the curve to the center of curvature at that point.

The position of the particle at any instant is defined by the distance, s, along the curve from a fixed reference point.

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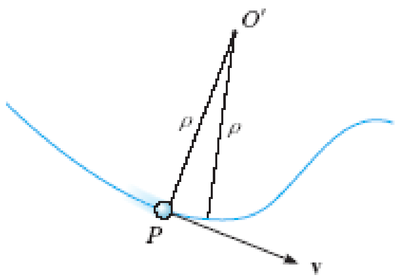
VELOCITY IN THE n-t COORDINATE SYSTEM



Velocity

The velocity vector is always tangent to the path of motion (t-direction).

The magnitude is determined by taking the time derivative of the path function, s(t).

$\boldsymbol{v} = v\boldsymbol{u}_t$ where $v = \dot{s} = ds/dt$

Here v <u>defines the magnitude</u> of the velocity (speed) and $\boldsymbol{u}_t$ <u>defines the direction</u> of the velocity vector.

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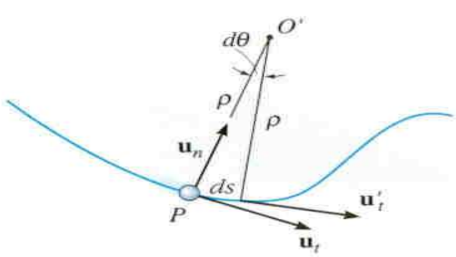
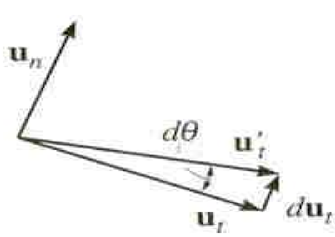
ACCELERATION IN THE n-t COORDINATE SYSTEM

Acceleration is the time rate of change of velocity:

$\boldsymbol{a} = d\boldsymbol{v}/dt = d(v\boldsymbol{u}_t)/dt = \dot{v}\boldsymbol{u}_t + v\dot{\boldsymbol{u}}_t$



Here $\dot{v}$ represents the change in the magnitude of velocity and $\dot{\boldsymbol{u}}_t$ represents the rate of change in the direction of $\boldsymbol{u}_t$.



After mathematical manipulation, the acceleration vector can be expressed as:

$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n.$

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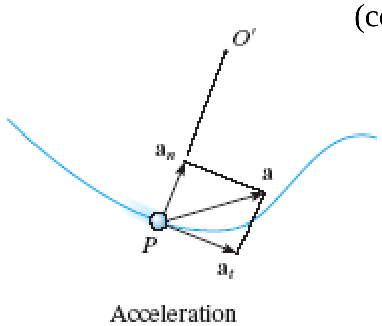
ACCELERATION IN THE n-t COORDINATE SYSTEM
(continued)



There are two components to the acceleration vector:

$\boldsymbol{a} = a_t\,\boldsymbol{u}_t + a_n\,\boldsymbol{u}_n$

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

  $a_t = \dot{v}$ or $a_t\,ds = v\,dv$

- The normal or centripetal component is always directed toward the center of curvature of the curve. $a_n = v^2/\rho$

- The magnitude of the acceleration vector is

  $a = [(a_t)^2 + (a_n)^2]^{0.5}$

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SPECIAL CASES OF MOTION

There are some special cases of motion to consider.

1) The particle moves along a straight line.

   $\rho \rightarrow \infty$ => $a_n = v^2/\rho = 0$ => $a = a_t = \dot{v}$

   The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.

   $a_t = \dot{v} = 0$ => $a = a_n = v^2/\rho$

   The normal component represents the time rate of change in the direction of the velocity.

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SPECIAL CASES OF MOTION
(continued)

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$. In this case,

$$s = s_o + v_o t + (1/2)(a_t)_c t^2$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2(a_t)_c(s - s_o)$$

As before, s_o and v_o are the initial position and velocity of the particle at t = 0. How are these equations related to projectile motion equations? Why?

4) The particle moves along a path expressed as y = f(x). The radius of curvature, ρ, at any point on the path can be calculated from

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{\left| d^2y/dx^2 \right|}$$

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THREE-DIMENSIONAL MOTION

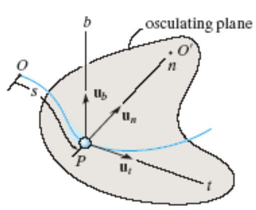

Fig. 12–26

If a particle moves along a space curve, the n and t axes are defined as before;

At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n and t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, $\boldsymbol{u}_b$, is directed perpendicular to the osculating plane, and its sense is defined by the cross product

$$\boldsymbol{u}_b = \boldsymbol{u}_t \times \boldsymbol{u}_n.$$

There is no motion, thus no velocity or acceleration, in the binomial direction.

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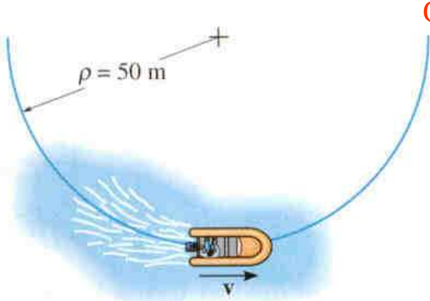
EXAMPLE PROBLEM



Given: Starting from rest, a motorboat travels around a circular path of ρ = 50 m at a speed that increases with time, $v = (0.2\ t^2)$ m/s.

Find: The magnitudes of the boat's velocity and acceleration at the instant t = 3 s.

Plan: The boat starts from rest (v = 0 when t = 0).

1) Calculate the velocity at t = 3s using v(t).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

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EXAMPLE
(continued)

Solution:

1) The velocity vector is $\mathbf{v} = v\,\mathbf{u}_t$, where the magnitude is given by $v = (0.2t^2)$ m/s. At t = 3s:

$$v = 0.2t^2 = 0.2(3)^2 = 1.8 \text{ m/s}$$

2) The acceleration vector is $\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n = \dot{v}\mathbf{u}_t + (v^2/\rho)\mathbf{u}_n$.

Tangential component: $a_t = \dot{v} = d(0.2t^2)/dt = 0.4t$ m/s^2
At t = 3s: $a_t = 0.4t = 0.4(3) = 1.2$ m/s^2

Normal component: $a_n = v^2/\rho = (0.2t^2)^2/(\rho)$ m/s^2
At t = 3s: $a_n = [(0.2)(3^2)]^2/(50) = 0.0648$ m/s^2

The magnitude of the acceleration is
$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(1.2)^2 + (0.0648)^2]^{0.5} = 1.20$ m/s^2

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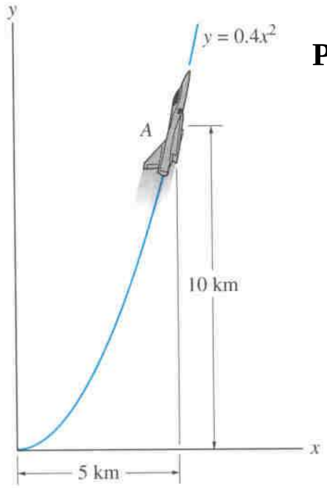

PROBLEM SOLVING

Given: A jet plane travels along a vertical parabolic path defined by the equation $y = 0.4x^2$. At point A, the jet has a speed of 200 m/s, which is increasing at the rate of 0.8 m/s^2.

Find: The magnitude of the plane's acceleration when it is at point A.

Plan:
1. The change in the speed of the plane (0.8 m/s^2) is the tangential component of the total acceleration.
2. Calculate the radius of curvature of the path at A.
3. Calculate the normal component of acceleration.
4. Determine the magnitude of the acceleration vector.

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PROBLEM SOLVING (continued)

Solution:

1) The tangential component of acceleration is the rate of increase of the plane's speed, so $a_t = \dot{v} = 0.8\ m/s^2$.

2) Determine the radius of curvature at point A (x = 5 km):

$dy/dx = d(0.4x^2)/dx = 0.8x$, $d^2y/dx^2 = d\,(0.8x)/dx = 0.8$
At x =5 km, $dy/dx = 0.8(5) = 4$, $d^2y/dx^2 = 0.8$

$$\Rightarrow \rho = \frac{[1 + (dy/dx)^2\ ]^{3/2}}{d^2y/dx^2} = [1 + (4)^2]^{3/2}/(0.8) = 87.62 \text{ km}$$

3) The normal component of acceleration is
$a_n = v^2/\rho = (200)^2/(87.62 \times 10^3) = 0.457\ m/s^2$

4) The magnitude of the acceleration vector is
$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(0.8)^2 + (0.457)^2]^{0.5} = 0.921\ m/s^2$

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CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

Objectives:

Students will be able to:

1. Determine velocity and acceleration components using cylindrical coordinates.

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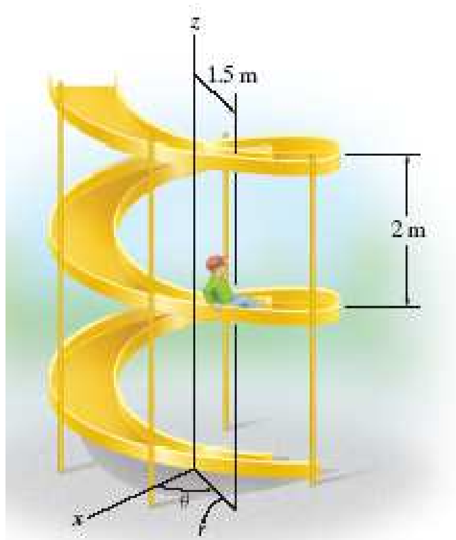
APPLICATIONS



The cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the boy slides down the slide at a constant speed of 2 m/s. How fast is his elevation from the ground changing (i.e., what is $\dot{z}$)?

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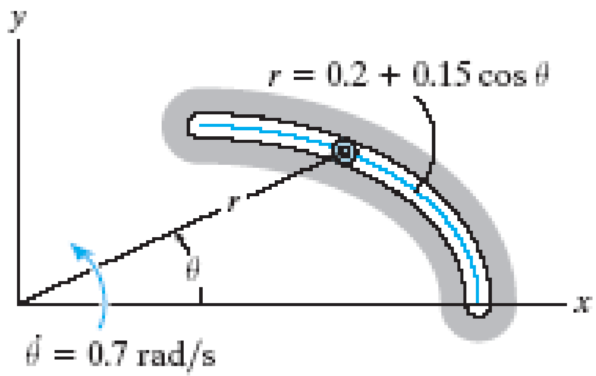
APPLICATIONS
(continued)



A polar coordinate system is a 2-D representation of the cylindrical coordinate system.

When the particle moves in a plane (2-D), and the radial distance, r, is not constant, the polar coordinate system can be used to express the path of motion of the particle.

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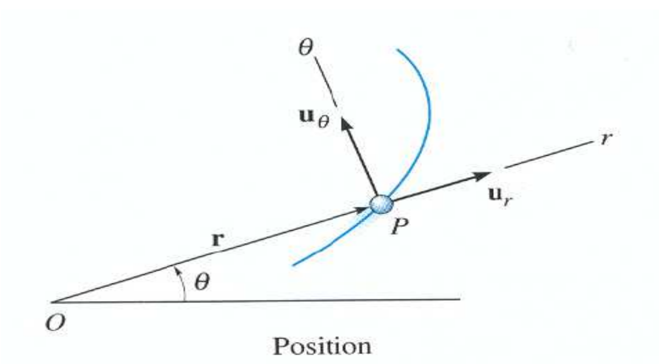
CYLINDRICAL COMPONENTS



We can express the location of P in polar coordinates as $\mathbf{r} = r\mathbf{u}_r$.
Note that:
The radial direction, r, extends outward from the fixed origin, O, and
The transverse coordinate, θ, is measured counter-clockwise (CCW) from the horizontal.

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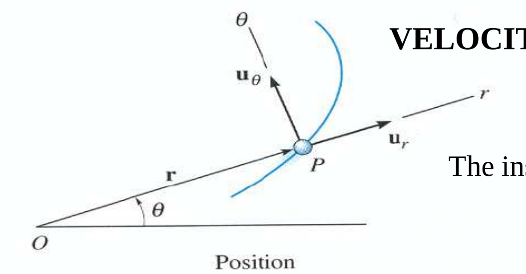
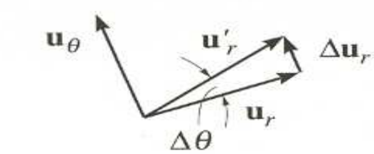
VELOCITY (POLAR COORDINATES)



The instantaneous velocity is defined as:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(r\boldsymbol{u}_r)/dt$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\,\frac{d\boldsymbol{u}_r}{dt}$$

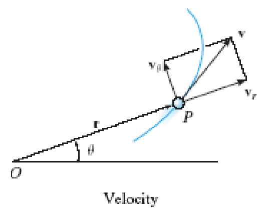


Using the chain rule:

$$d\boldsymbol{u}_r/dt = (d\boldsymbol{u}_r/d\theta)(d\theta/dt)$$

We can prove that $d\boldsymbol{u}_r/d\theta = \boldsymbol{u}_\theta$ so $d\boldsymbol{u}_r/dt = \dot{\theta}\boldsymbol{u}_\theta$

Therefore: $\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$



Thus, the velocity vector has two components:

1. $\dot{r}$, called the radial component,
2. $r\dot{\theta}$, called the transverse component.

The speed of the particle at any given instant is the sum of the squares of both components or

$$v = \sqrt{(r\,\dot{\theta}\,)^2 + (\,\dot{r}\,)^2}$$

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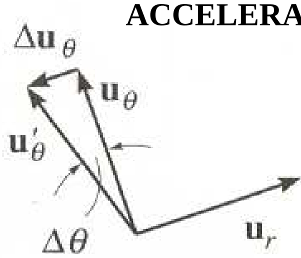
ACCELERATION (POLAR COORDINATES)

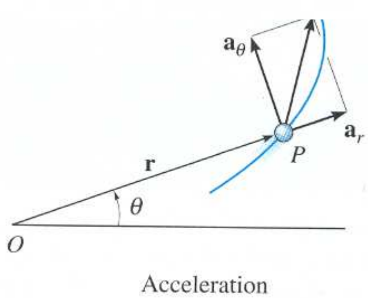


The instantaneous acceleration is defined as:

$$\boldsymbol{a} = d\boldsymbol{v}/dt = (d/dt)(\dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta)$$

After manipulation, the acceleration can be expressed as

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$



The term $(\ddot{r} - r\dot{\theta}^2)$ is the radial acceleration or a_r.

The term $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the transverse acceleration or a_θ

The magnitude of acceleration is $a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$

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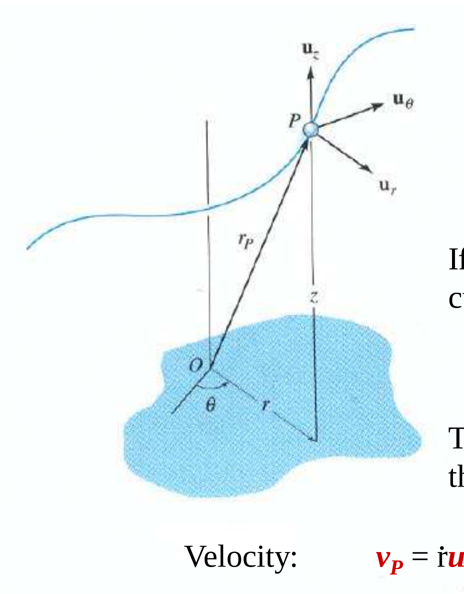
CYLINDRICAL COORDINATES



If the particle P moves along a space curve, its position can be written as

$$\mathbf{r}_P = r\mathbf{u}_r + z\mathbf{u}_z$$

Taking time derivatives and using the chain rule:

Velocity: $\mathbf{v}_P = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{u}_z$

Acceleration: $\mathbf{a}_P = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{u}_z$

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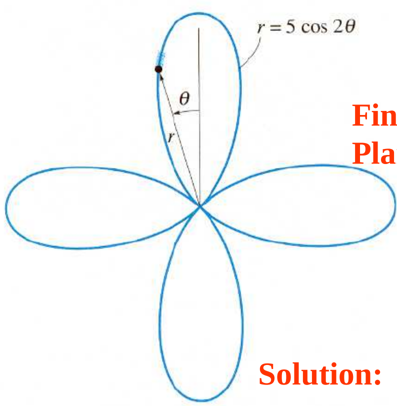
EXAMPLE



Given: $r = 5\cos(2\theta)$ (m)
$\dot{\theta} = 3t^2$ (rad/s)
$\theta_o = 0$

Find: Velocity and acceleration at $\theta = 30°$.

Plan: Apply chain rule to determine $\dot{r}$ and $\ddot{r}$ and evaluate at $\theta = 30°$.

Solution: $\theta = \int_{t_o=0}^{t} \dot{\theta}\, dt = \int_0^t 3t^2\, dt = t^3$

At $\theta = 30°$, $\theta = \frac{\pi}{6} = t^3$. Therefore: $t = 0.806$ s.

$\dot{\theta} = 3t^2 = 3(0.806)^2 = 1.95$ rad/s

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EXAMPLE
(continued)

$\ddot{\theta} = 6t = 6(0.806) = 4.836 \text{ rad/s}^2$

$r = 5\cos(2\theta) = 5\cos(60) = 2.5\text{m}$

$\dot{r} = -10\sin(2\theta)\dot{\theta} = -10\sin(60)(1.95) = -16.88 \text{ m/s}$

$\ddot{r} = -20\cos(2\theta)\dot{\theta}^2 - 10\sin(2\theta)\ddot{\theta}$

$= -20\cos(60)(1.95)^2 - 10\sin(60)(4.836) = -80 \text{ m/s}^2$

Substitute in the equation for velocity

$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$

$\boldsymbol{v} = -16.88\boldsymbol{u}_r + 2.5(1.95)\boldsymbol{u}_\theta$

$\boldsymbol{v} = \sqrt{(16.88)^2 + (4.87)^2} = 17.57 \text{ m/s}$

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EXAMPLE
(continued)

Substitute in the equation for acceleration:

$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$

$\boldsymbol{a} = [-80 - 2.5(1.95)^2]\boldsymbol{u}_r + [2.5(4.836) + 2(-16.88)(1.95)]\boldsymbol{u}_\theta$

$\boldsymbol{a} = -89.5\boldsymbol{u}_r - 53.7\boldsymbol{u}_\theta \text{ m/s}^2$

$a = \sqrt{(89.5)^2 + (53.7)^2} = 104.4 \text{ m/s}^2$

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