Carleton University * Faculty of Engineering and Design

Mechanics I

ECOR 1101

Lecture 17

CURVILINEAR MOTION:
GENERAL & RECTANGULAR COMPONENTS
MOTION OF A PROJECTILE

Section 9.4-9.6

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CURVILINEAR MOTION:
GENERAL & RECTANGULAR COMPONENTS

<u>Today's Objectives</u>:

Students will be able to:

1. Describe the motion of a particle traveling along a curved path.
2. Relate kinematic quantities in terms of the rectangular components of the vectors.

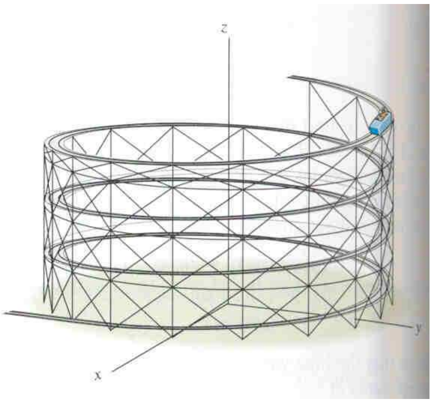
APPLICATIONS

The path of motion of each plane in this formation can be tracked with radar and their x, y, and z coordinates (relative to a point on earth) recorded as a function of time.

How can we determine the velocity or acceleration of each plane at any instant?

Should they be the same for each aircraft?

APPLICATIONS
(continued)



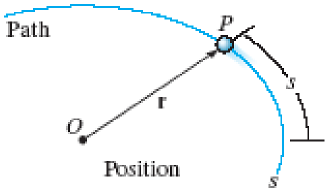
A roller coaster car travels down a fixed, helical path at a constant speed.

How can we determine its position or acceleration at any instant?

If you are designing the track, why is it important to be able to predict the acceleration ($F = ma$) of the car?

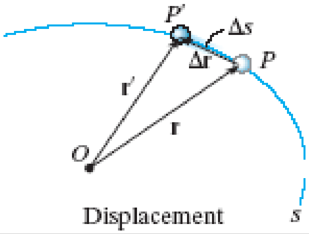
GENERAL CURVILINEAR MOTION
(Section 12.4)

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, vectors are used to describe the motion.



A particle moves along a curve defined by the path function, s.

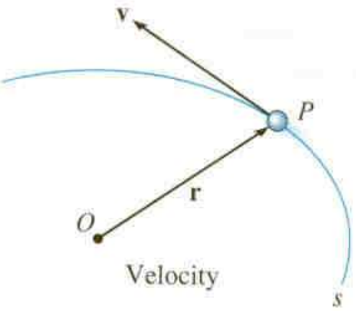
The position of the particle at any instant is designated by the vector $\boldsymbol{r} = \boldsymbol{r}(t)$. Both the magnitude and direction of $\boldsymbol{r}$ may vary with time.



If the particle moves a distance Δs along the curve during time interval Δt, the displacement is determined by vector subtraction: $\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}$

VELOCITY

Velocity represents the rate of change in the position of a particle.



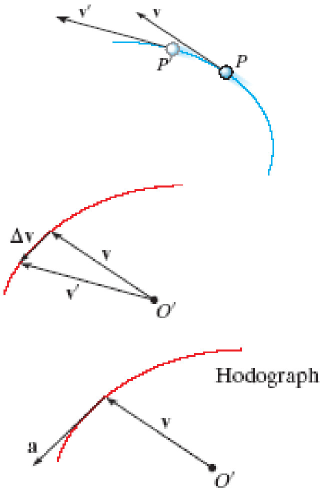
The average velocity of the particle during the time increment Δt is $\boldsymbol{v}_{avg} = \Delta \boldsymbol{r}/\Delta t$.

The instantaneous velocity is the time-derivative of position $\boldsymbol{v} = d\boldsymbol{r}/dt$.

The velocity vector, $\boldsymbol{v}$, is always tangent to the path of motion.

The magnitude of $\boldsymbol{v}$ is called the speed. Since the arc length Δs approaches the magnitude of $\Delta \boldsymbol{r}$ as $t \rightarrow 0$, the speed can be obtained by differentiating the path function ($v = ds/dt$). Note that this is not a vector!

ACCELERATION



Acceleration represents the rate of change in the velocity of a particle.

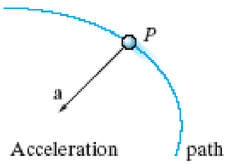
If a particle's velocity changes from $\mathbf{v}$ to $\mathbf{v}'$ over a time increment Δt, the average acceleration during that increment is:

$$\mathbf{a}_{avg} = \Delta\mathbf{v}/\Delta t = (\mathbf{v} - \mathbf{v}')/\Delta t$$

The instantaneous acceleration is the time-derivative of velocity:

$$\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$$

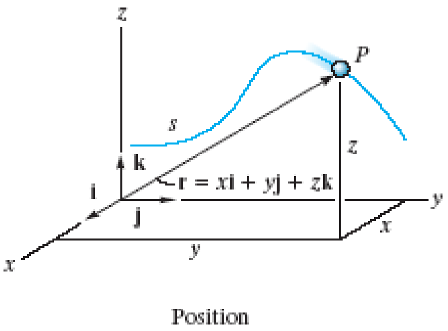
A plot of the locus of points defined by the arrowhead of the velocity vector is called a hodograph.



The acceleration vector is tangent to the hodograph, but not, in general, tangent to the path function.

CURVILINEAR MOTION: RECTANGULAR COMPONENTS
(Section 12.5)

It is often convenient to describe the motion of a particle in terms of its x, y, z or rectangular components, relative to a fixed frame of reference.



The position of the particle can be defined at any instant by the position vector

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\ .$$

The x, y, z components may all be functions of time, i.e., $x = x(t)$, $y = y(t)$, and $z = z(t)$.

The magnitude of the position vector is: $r = (x^2 + y^2 + z^2)^{0.5}$

The direction of $\mathbf{r}$ is defined by the unit vector: $\mathbf{u}_r = (\mathbf{r}/r)$

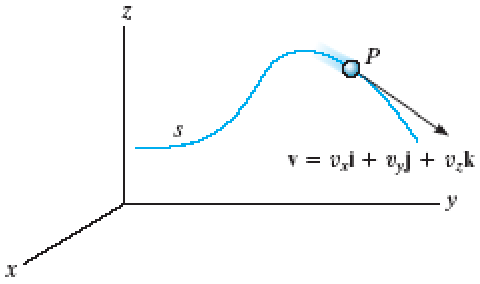
RECTANGULAR COMPONENTS: VELOCITY

The velocity vector is the time derivative of the position vector:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(x\boldsymbol{i})/dt + d(y\boldsymbol{j})/dt + d(z\boldsymbol{k})/dt$$

Since the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are constant in magnitude and direction, this equation reduces to $v = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$

where $v_x = \dot{x} = dx/dt$, $v_y = \dot{y} = dy/dt$, $v_z = \dot{z} = dz/dt$



Velocity

The magnitude of the velocity vector is

$$v = [(v_x)^2 + (v_y)^2 + (v_z)^2]^{0.5}$$

The direction of $\boldsymbol{v}$ is tangent to the path of motion.

RECTANGULAR COMPONENTS: ACCELERATION

The acceleration vector is the time derivative of the velocity vector (second derivative of the position vector):

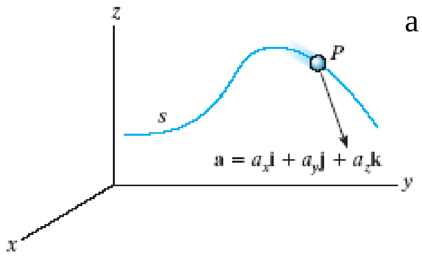
$$\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2 = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

where $a_x = \dot{v}_x = \ddot{x} = dv_x/dt$, $a_y = \dot{v}_y = \ddot{y} = dv_y/dt$,

$a_z = \dot{v}_z = \ddot{z} = dv_z/dt$

The magnitude of the acceleration vector is

$$a = [(a_x)^2 + (a_y)^2 + (a_z)^2]^{0.5}$$



Acceleration

The direction of $\boldsymbol{a}$ is usually not tangent to the path of the particle.

EXAMPLE

Given: The motion of two particles (A and B) is described by the position vectors
$\mathbf{r}_A = [3t\,\mathbf{i} + 9t(2 - t)\,\mathbf{j}]$ m
$\mathbf{r}_B = [3(t^2 - 2t + 2)\,\mathbf{i} + 3(t - 2)\,\mathbf{j}]$ m

Find: The point at which the particles collide and their speeds just before the collision.

Plan:
1) The particles will collide when their position vectors are equal, or $\mathbf{r}_A = \mathbf{r}_B$.
2) Their speeds can be determined by differentiating the position vectors.

EXAMPLE

(continued)

Solution:

1) The point of collision requires that $\mathbf{r}_A = \mathbf{r}_B$, so $x_A = x_B$ and $y_A = y_B$

x-components: $3t = 3(t^2 - 2t + 2)$
Simplifying: $t^2 - 3t + 2 = 0$
Solving: $t = \{3 \pm [3^2 - 4(1)(2)]^{0.5}\}/2(1)$
=> $t = 2$ or 1 s

y-components: $9t(2 - t) = 3(t - 2)$
Simplifying: $3t^2 - 5t - 2 = 0$
Solving: $t = \{5 \pm [5^2 - 4(3)(-2)]^{0.5}\}/2(3)$
=> $t = 2$ or $-1/3$ s

So, the particles collide when t = 2 s. Substituting this value into $\mathbf{r}_A$ or $\mathbf{r}_B$ yields

$x_A = x_B = 6$ m and $y_A = y_B = 0$

EXAMPLE

(continued)

2) Differentiate $\mathbf{r}_A$ and $\mathbf{r}_B$ to get the velocity vectors.

$\mathbf{v}_A = d\mathbf{r}_A/dt = \dot{x}_A\,\mathbf{i} + \dot{y}_A\,\mathbf{j} = [3\mathbf{i} + (18 - 18t)\mathbf{j}]$ m/s

At t = 2 s: $\mathbf{v}_A = [3\mathbf{i} - 18\mathbf{j}]$ m/s

$\mathbf{v}_B = d\mathbf{r}_B/dt = \dot{x}_B\mathbf{i} + \dot{y}_B\mathbf{j} = [(6t - 6)\mathbf{i} + 3\mathbf{j}]$ m/s

At t = 2 s: $\mathbf{v}_B = [6\mathbf{i} + 3\mathbf{j}]$ m/s

Speed is the magnitude of the velocity vector.

$v_A = (3^2 + 18^2)^{0.5} = 18.2$ m/s

$v_B = (6^2 + 3^2)^{0.5} = 6.71$ m/s

PROBLEM SOLVING

Given: A particle travels along a path described by the parabola $y = 0.5x^2$. The x-component of velocity is given by $v_x = (5t)$ ft/s. When t = 0, x = y = 0.

Find: The particle's distance from the origin and the magnitude of its acceleration when t = 1 s.

Plan: Note that v_x is given as a function of time.

1) Determine the x-component of position and acceleration by integrating and differentiating v_x, respectively.
2) Determine the y-component of position from the parabolic equation and differentiate to get a_y.
3) Determine the magnitudes of the position and acceleration vectors.

PROBLEM SOLVING (continued)

Solution:

1) x-components:

Velocity: $v_x = \dot{x} = dx/dt = (5t)$ ft/s

Position: $\int_0^x dx = \int_0^t 5t\,dt \Rightarrow x = (5/2)t^2 = (2.5t^2)$ ft

Acceleration: $a_x = \ddot{x} = \dot{v}_x = d/dt\,(5t) = 5$ ft/s^2

2) y-components:

Position: $y = 0.5x^2 = 0.5(2.5t^2)^2 = (3.125t^4)$ ft

Velocity: $v_y = dy/dt = d\,(3.125t^4)\,/dt = (12.5t^3)$ ft/s

Acceleration: $a_y = v_y = d\,(12.5t^3)\,/dt = (37.5t^2)$ ft/s^2

PROBLEM SOLVING (continued)

3) The distance from the origin is the magnitude of the position vector:

$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} = [2.5t^2\,\boldsymbol{i} + 3.125t^4\,\boldsymbol{j}]$ ft

At t = 1 s, $\boldsymbol{r} = (2.5\,\boldsymbol{i} + 3.125\,\boldsymbol{j})$ ft

Distance: $d = r = (2.5^2 + 3.125^2)^{0.5} = 4.0$ ft

The magnitude of the acceleration vector is calculated as:

Acceleration vector: $\boldsymbol{a} = [5\,\boldsymbol{i} + 37.5t^2\,\boldsymbol{j}]$ ft/s^2

Magnitude: $a = (5^2 + 37.5^2)^{0.5} = 37.8$ ft/s^2

MOTION OF A PROJECTILE

Objectives:

Students will be able to:

1. Analyze the free-flight motion of a projectile.

READING QUIZ

1. The downward acceleration of an object in free-flight motion is

A) zero

B) increasing with time

C) 9.81 m/s^2

D) 9.81 ft/s^2

2. The horizontal component of velocity remains __________ during a free-flight motion.

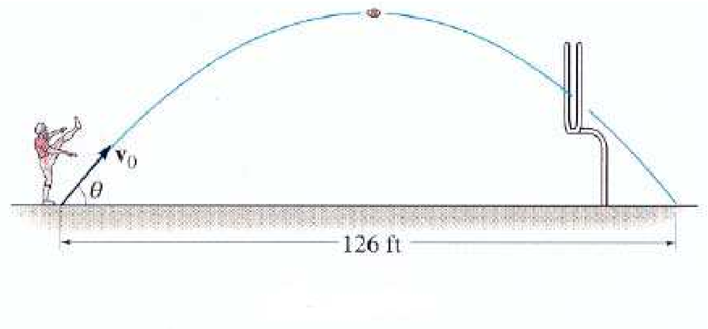
A) zero

B) constant

C) at 9.81 m/s^2

D) at 32.2 ft/s^2

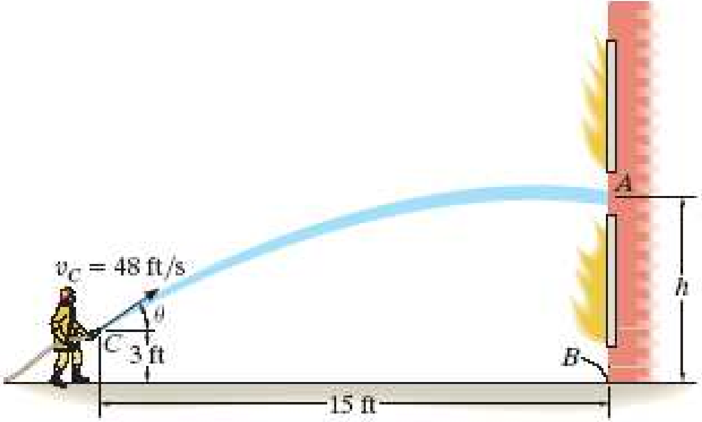
APPLICATIONS



A kicker should know at what angle, θ, and initial velocity, v_o, he must kick the ball to make a field goal.

For a given kick “strength”, at what angle should the ball be kicked to get the maximum distance?

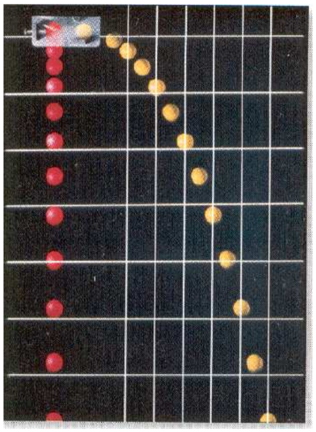
APPLICATIONS
(continued)



A fireman wishes to know the maximum height on the wall he can project water from the hose.
At what angle, θ, should he hold the hose?

MOTION OF A PROJECTILE
(Section 12.6)

Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in the vertical direction experiencing constant acceleration (i.e., gravity).



For illustration, consider the two balls on the left. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity. Each picture in this sequence is taken after the same time interval. Notice both balls are subjected to the same downward acceleration since they remain at the same elevation at any instant.

Also, note that the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction is constant.

KINEMATIC EQUATIONS: HORIZONTAL MOTION

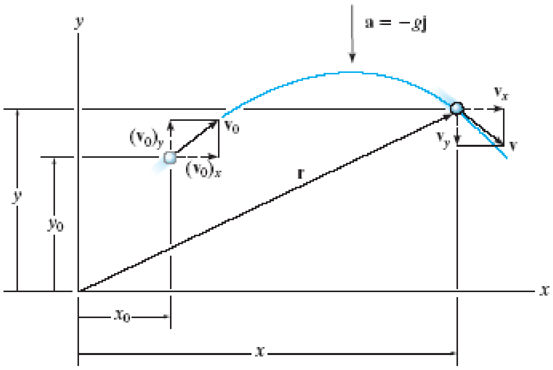


Fig. 12–20

Since $a_x = 0$, the velocity in the horizontal direction remains constant ($v_x = v_{ox}$) and the position in the x direction can be determined by:

$$x = x_o + (v_{ox})(t)$$

Why is a_x equal to zero (assuming movement through the air)?

KINEMATIC EQUATIONS: VERTICAL MOTION

Since the positive y-axis is directed upward, $a_y = -g$. Application of the constant acceleration equations yields:

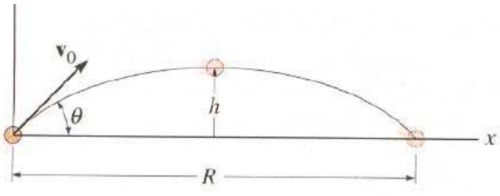
$$v_y = v_{oy} - g(t)$$

$$y = y_o + (v_{oy})(t) - \frac{1}{2}g(t)^2$$

$$v_y^2 = v_{oy}^2 - 2g(y - y_o)$$

For any given problem, only two of these three equations can be used.
Why?

EXAMPLE



Given: v_o and θ
Find: The equation that defines y as a function of x.
Plan: Eliminate time from the kinematic equations.

Solution: Using $v_x = v_o \cos \theta$ and $v_y = v_o \sin \theta$

We can write: $x = (v_o \cos \theta)t$ or $t = \dfrac{x}{v_o \cos \theta}$

$$y = (v_o \sin \theta)t - \frac{1}{2} g(t)^2$$

By substituting for t:

$$y = (v_o \sin \theta)\left(\frac{x}{v_o \cos \theta}\right) - \left(\frac{g}{2}\right)\left(\frac{x}{v_o \cos \theta}\right)^2$$

EXAMPLE
(continued)

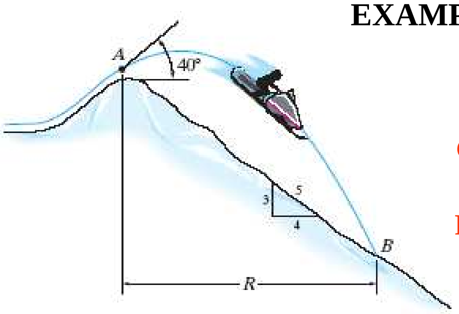
Simplifying the last equation, we get:

$$y = (x \tan\theta) - \left(\frac{g\,x^2}{2v_o^2}\right)(1 + \tan^2\theta)$$

The above equation is called the "path equation" which describes the path of a particle in projectile motion.

The equation shows that the path is parabolic.

EXAMPLE II



Given: Snowmobile is going 10 m/s at point A.

Find: The horizontal distance it travels (R) and the time in the air.

Solution:

First, place the coordinate system at point A. Then write the equation for horizontal motion.

$\rightarrow +$ $x_B = x_A + v_{Ax}t_{AB}$ and $v_{Ax} = 10 \cos 40^\circ$ m/s

Now write a vertical motion equation. Use the distance equation.

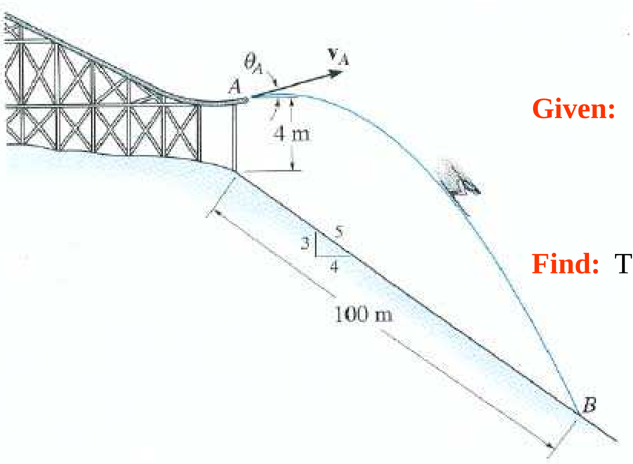
$\uparrow +$ $y_B = y_A + v_{Ay}t_{AB} - 0.5g_c t_{AB}^2$ $v_{Ay} = 10 \sin 40^\circ$ m/s

Note that $x_B = R$, $x_A = 0$, $y_B = -(3/4)R$, and $y_A = 0$.

Solving the two equations together (two unknowns) yields

$R = 19.0$ m $t_{AB} = 2.48$ s

PROBLEM SOLVING



Given: Skier leaves the ramp at $\theta_A = 25^o$ and hits the slope at B.

Find: The skier's initial speed v_A.

Plan: Establish a fixed x,y coordinate system (in the solution here, the origin of the coordinate system is placed at A). Apply the kinematic relations in x and y-directions.

PROBLEM SOLVING (continued)

Solution:

Motion in x-direction:

Using $x_B = x_A + v_{ox}(t_{AB})$

$$t_{AB} = \frac{(4/5)100}{v_A\,(\cos 25)} = \frac{88.27}{v_A}$$

Motion in y-direction:

Using $y_B = y_A + v_{oy}(t_{AB}) - ½\, g(t_{AB})^2$

$$-(60+4) = 0 + v_A(\sin 25)\left(\frac{80}{v_A\,(\cos 25)}\right) - ½\,(9.81)\left(\frac{88.27}{v_A}\right)^2$$

$$\underline{\underline{v_A = 19.42\text{ m/s}}}$$