Student number:____________________, Total marks: ___________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [4 points] Consider the matrix

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}.$$

(a) Show that 3 + 2i is an eigenvalue of A.
(b) Find the other eigenvalue.
(c) Calculate an eigenvector corresponding to 3 + 2i.

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)+4 = \lambda^2-6\lambda+13$$

$$\lambda = \frac{1}{2}\left(6 \pm \sqrt{36-52}\right) = \frac{1}{2}\left(6 \pm \sqrt{-16}\right) = 3 \pm 2i$$

$\rightarrow$ The eigenvalues are $3 \pm 2i$

$$(A-(3+2i)I)v=0 \rightarrow \begin{bmatrix} 3-3-2i & -2 \\ 2 & 3-3-2i \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$R_1 + iR_2$

$$\rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} i \\ 1 \end{bmatrix} t$$

$t \neq 0$

Question 2. [3 points] Calculate the eigenvalues (not the eigenvectors) of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 5 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 5 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)(1-\lambda)(4-\lambda) - 2(1-\lambda)$$

$$= (1-\lambda)\left[(3-\lambda)(4-\lambda) - 2\right]$$

$$= (1-\lambda)\left[\lambda^2 - 7\lambda + 10\right]$$

$$= (1-\lambda)(\lambda - 2)(\lambda - 5)$$

$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 5$

Question 3. [3 points] The matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

has $\lambda = 1$ as an eigenvalue. Calculate a corresponding eigenvector.

$(A - 1 \cdot I)v = 0$ Find $v \neq 0$.

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_3 = s$ is free

$v_2 = -2s$

$v_1 = v_2 + v_3 = -2s + s = -s$

$$\Rightarrow \quad v = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} s \qquad s \neq 0$$

Question 4. [6 points] Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \\ 0 & 2 & -5 \end{bmatrix}.$$

(a) Find the inverse of A.

(b) Find the solution of $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -2 & 1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \xrightarrow[+6R_3+R_2]{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & -3 \\ 0 & 2 & 0 & 10 & -5 & 6 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & -3 \\ 0 & 1 & 0 & 5 & -\frac{5}{2} & 3 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -5 & 3 & -3 \\ 5 & -\frac{5}{2} & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \to x = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{11}{2} \\ 2 \end{bmatrix}$$

Question 5. [4 points] (a) For which values of a, b does the system

$$x + ay = 3, \qquad 2x + 3y = b$$

have
(a) no solution?
(b) a unique solution?
(c) infinitely many solutions?

$$\left[\begin{array}{cc|c} 1 & a & 2 \\ 2 & 3 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 3-2a & b-4 \end{array}\right] \quad -2R_1 + R_2$$

unique solution: $3-2a \neq 0$ or $a \neq \frac{3}{2}$

no solution: $a = \frac{3}{2}$ and $b \neq 4$

infinitely many solutions: $a = \frac{3}{2}$ and $b = 4$

Question 6. [6 points] Consider the system of differential equations

$$x' = 4(y - x), \qquad y' = x - y.$$

(a) Calculate the eigenvalues and corresponding eigenvectors of the coefficient matrix.
(b) Give the general solution of this differential equation.
(c) Find the unique solution with initial conditions $x(0) = 6, y(0) = 1$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = (-4-\lambda)(-1-\lambda) - 4$$

$$= \lambda^2 + 5\lambda = \lambda(\lambda+5)$$

$$\lambda_1 = 0, \qquad \lambda_2 = -5$$

$$\lambda_1 = 0 : \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix} v = 0 \to v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s \qquad s \neq 0$$

$$\lambda_2 = -5 : \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} w = 0 \to w = \begin{bmatrix} 4 \\ -1 \end{bmatrix} s \qquad s \neq 0$$

General solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t} v + c_2 e^{\lambda_2 t} w = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-5t} \begin{bmatrix} 4 \\ -1 \end{bmatrix} c_2$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 6 \\ 1 \end{bmatrix} \to c_1 = 2, \; c_2 = 1$$

Unique solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{-5t} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Question 7. [4 points] Bobby the bird lives on Hawaii, where he travels between the islands of Maui (M) and Big Island (B). People tell you that Bobby's movement between M and B can be modeled as a Markov chain with the transition matrix

$$P = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix},$$

where 0.6 is the probability that Bobby will stay on Maui from one week to the next.
(a) If Bobby is on Maui this week, what is the probability that he is on Maui in two weeks?
(b) What is the percentage of time that Bobby spends on Maui in the long run?

a) $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Bobby in Maui this week

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = P \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.36 + 0.2 \\ 0.24 + 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$

$\rightarrow$ Bobby is on Maui in two weeks with prob. 0.56

b) $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = P \begin{bmatrix} x_n \\ y_n \end{bmatrix} \rightarrow (P - I)\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 0$

$\begin{bmatrix} -0.4 & 0.5 \\ 0.4 & -0.5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \cdot s \qquad s \neq 0$

For Probabilities: $x_n + y_n = 1 \rightarrow 5s + 4s = 1$

$\rightarrow s = \frac{1}{9}$

In the long run, Bobby spends $\frac{5}{9} \cdot 100\%$ on Maui

Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [4 points] Consider the matrix

$$A = \begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}.$$

(a) Show that 5 + 2i is an eigenvalue of A.
(b) Find the other eigenvalue.
(c) Calculate an eigenvector corresponding to 5 + 2i.

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda) + 4 = \lambda^2 - 10\lambda + 29$$

$$\lambda_{1/2} = \frac{1}{2}\left(10 \pm \sqrt{100 - 116}\right) = \frac{1}{2}\left(10 \pm \sqrt{-16}\right) = 5 \pm 2i$$

The eigenvalues are $\lambda_{1/2} = 5 \pm 2i$

$$\left(A - (5+2i)I\right) = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \longrightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ -i \end{bmatrix} s \qquad s \neq 0$$

Question 2. [3 points] Calculate the eigenvalues (not the eigenvectors) of the matrix

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ -1 & 5 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ 0 & -1-\lambda & 0 \\ -1 & 5 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda)(4-\lambda) - 2(-1-\lambda) = (-1-\lambda)\left[(3-\lambda)(4-\lambda) - 2\right]$$

$$= (-1-\lambda)\left[\lambda^2 - 7\lambda + 10\right] = (-1-\lambda)(\lambda-2)(\lambda-5)$$

$$\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 5$$

Question 3. [3 points] The matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

has $\lambda = 2$ as an eigenvalue. Calculate a corresponding eigenvector.

$(A - 2I)v = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_2 = s$ is free

$v_3 = 0$

$v_1 = v_2 - v_3 = s$

$$v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} s \qquad s \neq 0$$

Question 4. [6 points] Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 0 \\ 0 & 4 & -3 \end{bmatrix}.$$

(a) Find the inverse of A.

(b) Find the solution of $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 4 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -2 & 1 & 0 \\ 0 & 4 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \xrightarrow[4R_3+R_2]{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 4 & 0 & 6 & -3 & 4 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 1 & 0 & 3/2 & -3/4 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -3 & 2 & -2 \\ 3/2 & -3/4 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \to x = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7/4 \\ 2 \end{bmatrix}$$

Question 5. [4 points] (a) For which values of a, b does the system

$$x + ay = 3, \qquad 3x + 7y = b$$

have
(a) no solution?
(b) a unique solution?
(c) infinitely many solutions?

$$\left[\begin{array}{cc|c} 1 & a & 3 \\ 3 & 7 & b \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{cc|c} 1 & a & 3 \\ 0 & 7-3a & b-9 \end{array}\right]$$

unique solution: $7 - 3a \neq 0$ or $a \neq \frac{7}{3}$

no solution: $a = \frac{7}{3}$ and $b \neq 9$

infinitely many solutions: $a = \frac{7}{3}$ and $b = 9$

Question 6. [6 points] Consider the system of differential equations

$$x' = 5(y - x), \qquad y' = x - y.$$

(a) Calculate the eigenvalues and corresponding eigenvectors of the coefficient matrix.
(b) Give the general solution of this differential equation.
(c) Find the unique solution with initial conditions $x(0) = 7, y(0) = 1$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix} = (-5-\lambda)(-1-\lambda) - 5 = \lambda^2 + 6\lambda$$
$$= \lambda(\lambda+6)$$

$$\lambda_1 = 0 \rightarrow Av = 0 \rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s$$

$$s \neq 0$$

$$\lambda_2 = -6 \rightarrow (A + 6I)w = 0 \rightarrow \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} w = 0 \qquad w = \begin{bmatrix} 5 \\ -1 \end{bmatrix} s$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{\lambda_1 t} v + c_2 e^{\lambda_2 t} w = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} \Rightarrow c_1 = 2, \ c_2 = 1$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{-6t} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Question 7. [4 points] Bobby the bird lives on Hawaii, where he travels between the islands of Maui (M) and Big Island (B). People tell you that Bobby's movement between M and B can be modeled as a Markov chain with the transition matrix

$$P = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix},$$

where 0.7 is the probability that Bobby will stay on Maui from one week to the next.
(a) If Bobby is on Maui this week, what is the probability that he is on Maui in two weeks?
(b) What is the percentage of time that Bobby spends on Maui in the long run?

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Bobby is on Maui in week zero.

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = P^2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.49 + 0.06 \\ 0.21 + 0.24 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$$

In week two, Bobby is on Maui with probability 0.55

In the long run $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = P \begin{bmatrix} x^* \\ y^* \end{bmatrix}$ or $(P - I)\begin{bmatrix} x^* \\ y^* \end{bmatrix} = 0$

$$\begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix} = 0 \qquad \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} s \qquad s \neq 0$$

Probabilities: $x^* + y^* = 1 \to 2s + 3s = 1 \to s = \frac{1}{5}$

In the long run, Bobby spends $\frac{2}{5} \cdot 100\% = 40\%$ of his time on Maui.

Student number:______________________, Total marks: ____________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [4 points] Consider the matrix

$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

(a) Show that 3 + 4i is an eigenvalue of A.
(b) Find the other eigenvalue.
(c) Calculate an eigenvector corresponding to 3 + 4i.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)+4 = \lambda^2 - 6\lambda + 25$$

$$\lambda = \frac{1}{2}\left(6 \pm \sqrt{36 - 100}\right) = \frac{1}{2}\left(6 \pm \sqrt{-64}\right) = 3 \pm 4i$$

The eigenvalues are $\lambda = 3 \pm 4i$

$$(A - (3+4i)I) = \begin{bmatrix} 3-(3+4i) & -4 \\ 4 & 3-(3+4i) \end{bmatrix} = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \to \begin{bmatrix} -4i & -4 \\ 0 & 0 \end{bmatrix}$$

So $v = \begin{bmatrix} 1 \\ -i \end{bmatrix} s \quad s \neq 0$ is an eigenvector

Question 2. [3 points] Calculate the eigenvalues (not the eigenvectors) of the matrix

$$A = \begin{bmatrix} -3 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & -5 & -4 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & -5 & -4-\lambda \end{vmatrix} = (-3-\lambda)(1-\lambda)(-4-\lambda) - 2(1-\lambda)$$

$$= (1-\lambda)\left[(-3-\lambda)(-4-\lambda) - 2\right] = (1-\lambda)\left[\lambda^2 + 7\lambda + 10\right]$$

$$= (1-\lambda)(\lambda+2)(\lambda+5)$$

The eigenvalues are $\lambda = 1$, $\lambda = -2$, $\lambda = -5$

Question 3. [3 points] The matrix

$$A = \begin{bmatrix} 5 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

has $\lambda = 2$ as an eigenvalue. Calculate a corresponding eigenvector.

$$(A-2I) = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} \xrightarrow[3R_1+R_2]{2R_1+R_2} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_3 = s$

$v_2 = -2s$

$v_1 = v_2 + v_3 = -2s + s = -s$

$$v = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} s \qquad s \neq 0$$

Question 4. [6 points] Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 4 & 0 \\ 0 & 4 & -7 \end{bmatrix}.$$

(a) Find the inverse of A.

(b) Find the solution of $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 4 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 4 & -8 & -2 & 1 & 0 \\ 0 & 4 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2 + R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 4 & -8 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \xrightarrow[8R_3 + R_2]{-4R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 4 & -4 \\ 0 & 4 & 0 & 14 & -7 & 8 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 4 & -4 \\ 0 & 1 & 0 & \frac{7}{2} & -\frac{7}{4} & 2 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -7 & 4 & -4 \\ 7/2 & -7/4 & 2 \\ 2 & -1 & 1 \end{bmatrix}$$

$$Ax = b \to x = A^{-1}b = \begin{bmatrix} -7 \\ 15/4 \\ 2 \end{bmatrix}$$

Question 5. [4 points] (a) For which values of a, b does the system

$$x + ay = 3, \qquad -2x + 5y = b$$

have
(a) no solution?
(b) a unique solution?
(c) infinitely many solutions?

$$\left[\begin{array}{cc|c} 1 & a & 3 \\ -2 & 5 & b \end{array}\right] \xrightarrow{2R_1 + R_2} \left[\begin{array}{cc|c} 1 & a & 3 \\ 0 & 5+2a & b+6 \end{array}\right]$$

if $5 + 2a \neq 0$ i.e. $a \neq -\frac{5}{2}$ unique solution

if $a = -\frac{5}{2}$ and $b \neq -6$ no solution

if $a = -\frac{5}{2}$ and $b = 6$ infinitely many solutions

Question 6. [6 points] Consider the system of differential equations

$$x' = 2(y - x), \qquad y' = x - y.$$

(a) Calculate the eigenvalues and corresponding eigenvectors of the coefficient matrix.
(b) Give the general solution of this differential equation.
(c) Find the unique solution with initial conditions $x(0) = 4, y(0) = 1$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = (-2-\lambda)(-1-\lambda) - 2 = \lambda^2 + 3\lambda = \lambda(\lambda+3)$$

$$\lambda_1 = 0 \qquad Av = 0 \quad \to \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s \qquad s \neq 0$$

$$\lambda_2 = -3 \qquad (A - 3I)w = 0 \quad \to \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} w = 0 \qquad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix} s$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t} v + c_2 e^{\lambda_2 t} w = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \Rightarrow \quad c_1 = 2, \ c_2 = 1$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Question 7. [4 points] Bobby the bird lives on Hawaii, where he travels between the islands of Maui (M) and Big Island (B). People tell you that Bobby's movement between M and B can be modeled as a Markov chain with the transition matrix

$$P = \begin{bmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{bmatrix},$$

where 0.1 is the probability that Bobby will stay on Maui from one week to the next.
(a) If Bobby is on Maui this week, what is the probability that he is on Maui in two weeks?
(b) What is the percentage of time that Bobby spends on Maui in the long run?

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Bobby on Maui in week 0

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = P^2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.46 \\ 0.54 \end{bmatrix}$

With probability 0.46 Bobby is on Maui in two weeks

Long run: $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = P \begin{bmatrix} x^* \\ y^* \end{bmatrix} \leftarrow (P - I) \begin{bmatrix} x^* \\ y^* \end{bmatrix} = 0$

$\begin{bmatrix} -0.9 & 0.5 \\ 0.9 & -0.5 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix} = 0 \qquad \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} s \qquad s \neq 0$

For probabilities: $x^* + y^* = 1$ so $5s + 9s = 1$ $s = \frac{1}{14}$

In the long run the percentage on Maui is $\frac{5}{14} \cdot 100\%$.