MAT 1332, Winter 2014, Assignment 7
This assignment will not be marked. It serves as preparation for the final exam.

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QUESTION 1. In each case, find the nullclines and steady states. Draw them in the phase plane and add the direction arrows. Then draw a typical solution curve in the phase plane and also as a function of time. For each steady state, find the Jacobi matrix and calculate the eigenvalues. Is the steady state stable?

a) Predator Prey

$$\frac{db}{dt} = 5(1-2p)b, \qquad \frac{dp}{dt} = (-2+3b)p.$$

b) Predator Prey with logistic prey growth

$$\frac{db}{dt} = (1-b)b - 3pb, \qquad \frac{dp}{dt} = -p/2 + pb.$$

c) Predator Prey with Holling type II functional response

$$\frac{db}{dt} = b/2 - \frac{b}{1+b}p, \qquad \frac{dp}{dt} = -p/3 + \frac{b}{1+b}p.$$

d) Competition (competitive exclusion)

$$\frac{da}{dt} = 3(1-(a+b/2))a, \qquad \frac{db}{dt} = 5(1-(3a+b))b.$$

e) Competition (coexistence)

$$\frac{da}{dt} = 3(1-(a+b/2))a, \qquad \frac{db}{dt} = 5(1-(a/3+b))b.$$

f) An infectious disease with permanent immunity

$$\frac{dS}{dt} = -2SI, \qquad \frac{dI}{dt} = 2SI - I.$$

g) An infectious disease with return to susceptible state

$$\frac{dS}{dt} = -2SI + I, \qquad \frac{dI}{dt} = 2SI - I.$$

QUESTION 2. Consider the following competition system

$$\frac{dx}{dt} = (1 - x - y/2)x, \quad \frac{dy}{dt} = (1 - x/3 - y)y.$$

Find the nullclines and the steady states, draw them in the phase plane and add direction arrows on the nullclines. Is there a coexistence steady state? Is it stable or unstable? Check the direction arrows to answer this question. Then calculate the Jacobi matrix at the coexistence point and find its eigenvalues to calculate stability.

Solution:

a) We need to solve

$$\begin{aligned} \frac{d}{dt}x &= 0 = \quad x - x^2 - \frac{xy}{2} \\ \frac{d}{dt}y &= 0 = \quad y - y^2 - \frac{xy}{3}. \end{aligned}$$

Factor x in the first equation to get $x(1 - x - \frac{y}{2}) = 0$. Hence $x = 0$ or $1 - x - y/2 \Rightarrow y = 2 - 2x$.
Similarly, from the second equation we get $y(1 - y - x/3)$, and so $y = 0$ or $1 - y - x/3 = 0 \Rightarrow y = 1 - x/3$.
Now, the first two lines (from the first equation) intersect with the other two (from the second equation) at exactly four points. In fact, we have the four linear systems

$$\begin{aligned} x &= 0 \\ y &= 0; \end{aligned}$$

$$\begin{aligned} x &= 0 \\ y + x/3 &= 1; \end{aligned}$$

$$\begin{aligned} 2x + y &= 2 \\ y &= 0; \end{aligned}$$

$$\begin{aligned} 2x + y &= 2 \\ y + x/3 &= 1. \end{aligned}$$

These systems have (each) a unique solution, which yields the following four intersection points

$$(0,0);\ (0,1);\ (1,0);\ (\frac{3}{5}, \frac{4}{5}).$$

b) We have the following phase diagram.

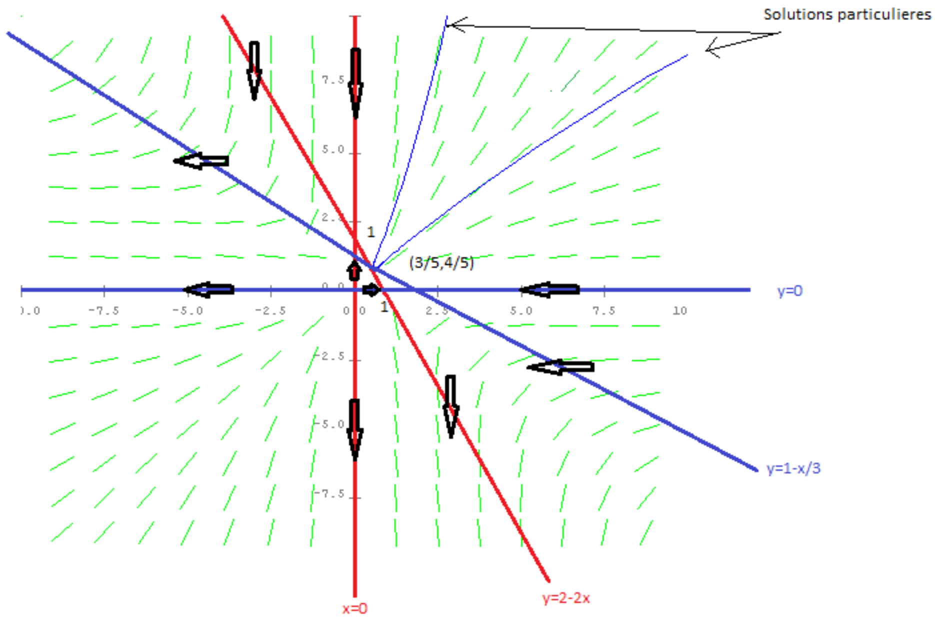


Figure 1: Phase portrait of the system: $x' = (1 - x - y/2)x, \quad y' = (1 - x/3 - y)y$

c) The coexistence steady state is $(3/5, 4/5)$. Since the arrows in the phase diagram are directed toward it is therefore stable.

d) The Jacobi matrix of this system is

$$J(x,y) = \begin{bmatrix} 1 - 2x - \frac{y}{2} & -\frac{x}{2} \\ -\frac{y}{3} & 1 - 2y - \frac{x}{3} \end{bmatrix}.$$

e) The coexistence steady state is $(\frac{3}{5}, \frac{4}{5})$. At this point we have

$$J(3/5, 4/5) = \begin{bmatrix} -\frac{3}{5} & -\frac{3}{10} \\ -\frac{4}{15} & -\frac{4}{5} \end{bmatrix}.$$

The eigenvalues of $J(3/5, 4/5)$ are obtained from the equation

$$(\lambda + \frac{2}{5})(\lambda + 1) = 0\,,$$

and hence the eigenvalues are -1 and $-2/5$.
Since the eigenvalues are both negative, this point is stable.

QUESTION 3. Consider a disease that propogates according to the system

$$\begin{aligned}\frac{dx}{dt} &= 20 - 5xy - 5x \\ \frac{dy}{dt} &= 5xy - 10y\end{aligned}$$

where x represents susceptible individuals, y represents infected individuals.

(a) Find the two biologically meaningful steady states.
Solution:
For equilibria, the equations satisfy

$$\begin{aligned}20 - 5xy - 5x = 0 \\ 5xy - 10y = 0\end{aligned}$$

From the second equation, either $y = 0$ or $x = 2$. When $y = 0$, the first equation tells us that $x = 4$. When $x = 2$, the first equation tells us that $y = 1$. Thus there are two equilibria, $(2, 1)$ and $(4, 0)$. Both are biologically meaningful. (Note that, for instance, $(2, 0)$ is not an equilibrium. Why?)

(b) Show that the Jacobian matrix of this system is given by

$$\begin{bmatrix} -5 - 5y & -5x \\ 5y & 5x - 10 \end{bmatrix}$$

Solution:
The Jacobian matrix of

$$\begin{aligned}x' = f_1(x, y) \\ y' = f_2(x, y)\end{aligned}$$

is the matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -5 - 5y & -5x \\ 5y & 5x - 10 \end{bmatrix}$$

(c) For each of the two steady states from (a) find the eigenvalues of the Jacobian matrix.
Solution:
For $(4, 0)$, we have

$$\det J(4, 0) = \begin{bmatrix} -5 & -20 \\ 0 & 10 \end{bmatrix}$$

Since this matrix is diagonal, the eigenvalues are -5 and 10.

For $(2,1)$, we have

$$\det J(2,1) = \begin{bmatrix} -10 & -10 \\ 5 & 0 \end{bmatrix}$$

$$\det(J(2,1) - \lambda I) = \begin{bmatrix} -10-\lambda & -10 \\ 5 & -\lambda \end{bmatrix}$$

$$= \lambda^2 + 10\lambda + 50$$

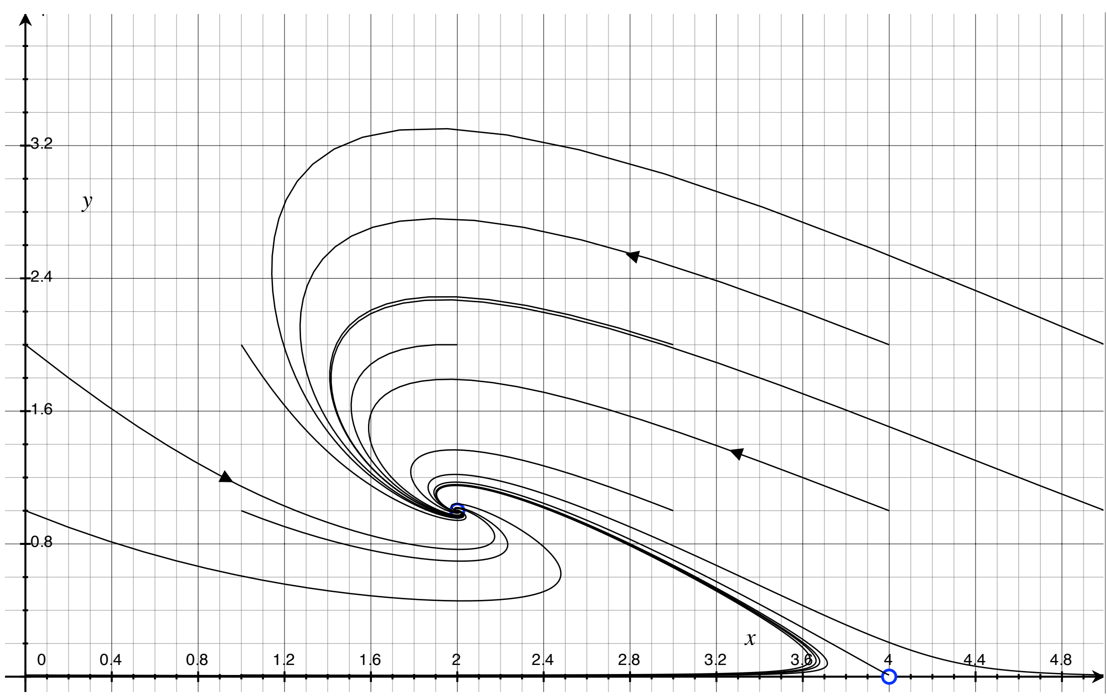
The eigenvalues are thus $\lambda = -5 \pm 5i$

(d) Determine the stability of the two steady states.

Solution:

The equilibrium $(4,0)$ is an unstable saddle. The equilibrium $(2,1)$ is a stable spiral.

(e) Draw solutions in the phase plane.



QUESTION 4. The following is a model for a pair-formation process in an open population. We denote by x the number of singles in the population and by y the number of pairs, both in thousands. When a pair forms, two singles disappear, and vice versa, when a pair splits up, two singles appear. In addition, singles and pairs immigrate from elsewhere, individuals die, and pairs have offspring, which are of course singles. The system of equations reads

$$\begin{aligned} \frac{d}{dt}x &= -2x + 2y + 4 \\ \frac{d}{dt}y &= x - 3y + 2. \end{aligned}$$

- How many singles and pairs are there at the steady state?
- Draw the nullclines in the phase plane.
- On each nullcline and in each of the four regions in phase space between the nullclines, draw the direction arrows.
- From your direction arrows, tell whether the steady state is stable or not. [Do not use linearization and Jacobi-matrix techniques here.]

Solution:

- In order to find the steady state we must find any ordered pair (x, y) such that $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

  The x-nullcline:

$$\begin{aligned} \frac{dx}{dt} &= 0 \\ -2x + 2y + 4 &= 0 \\ y &= x - 2 \end{aligned}$$

  The y-nullcline:

$$\begin{aligned} \frac{dy}{dt} &= 0 \\ x - 3y + 2 &= 0 \\ y &= \frac{1}{3}x + \frac{2}{3} \end{aligned}$$

  The steady states will be found at the intersections of the nullclines:

$$\begin{aligned} x - 2 &= \frac{1}{3}x + \frac{2}{3} \\ 3x - 6 &= x + 2 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

Since $y = (4) - 2 = 2$, the point $(4, 2)$ is a steady state. It has 4000 singles and 2000 pairs.

Consider $\frac{dx}{dt}$ on the y-nullcline:

$$\begin{aligned} \frac{dx}{dt} &= -2x + 2\left(\frac{1}{3}x + \frac{2}{3}\right) + 4 \\ &= -2x + \frac{2}{3}x + \frac{4}{3} + 4 \\ &= -\frac{4}{3}x + \frac{16}{3} \end{aligned}$$

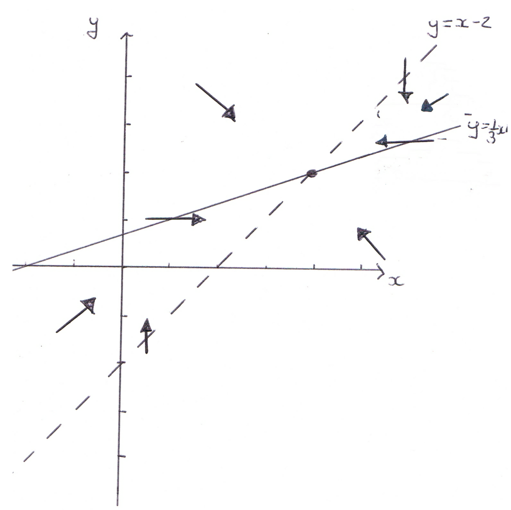
Hence, $\frac{dx}{dt}$ is positive when $x < 4$ and negative when $x > 4$.

Consider $\frac{dy}{dt}$ on the x-nullcline:

$$\begin{aligned} \frac{dy}{dt} &= x - 3(x - 2) + 2 \\ &= x - 3x + 6 + 2 \\ &= -2x + 8 \end{aligned}$$

Hence, $\frac{dy}{dt}$ is positive when $x < 4$ and negative when $x > 4$.

Therefore, the phase plane can be represented by :



Hence, we can see that the equilibrium is stable from the phase plane.

Question 5. The following system describes a predator prey system in which the prey has an Allee effect. What is the threshold of the prey to persist when alone? Find the nullclines and the steady states of the system. For which values of m is there a coexistence steady state? Draw the phase plane with direction arrows for $m = 0.9$. Sketch the solution curve starting at $(1.1, 0.1)$ and sketch each component of the solution as a function of time.

$$\frac{dx}{dt} = x(1-x)(x-0.5) - xy/8, \quad \frac{dy}{dt} = -my + xy.$$

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<u>Question 1c:</u>

$$\frac{db}{dt} = b/2 - \frac{b}{1+b}p \qquad \frac{dp}{dt} = -p/3 + \frac{b}{1+b}p$$

1) b-nullcline: set $\frac{db}{dt} = 0$ or $\frac{b}{2} - \frac{b}{1+b}p = 0$

Solutions: $b = 0$ or $p = \frac{1+b}{2}$

2) p-nullcline: set $\frac{dp}{dt} = 0$ or $-p/3 + \frac{b}{1+b}p = 0$

Solution: $p = 0$ or $\frac{b}{1+b} = \frac{1}{3}$ i.e. $p = 0$ or $b = \frac{1}{2}$

3) Steady states: $(b=0, p=0)$ or $\left(b = \frac{1}{2}, p = \frac{3}{4}\right)$

4) Jacobi matrix:

$$J = \begin{bmatrix} \frac{1}{2} - \frac{p}{(1+b)^2} & -\frac{b}{1+b} \\ \frac{p}{(1+b)^2} & -\frac{1}{3} + \frac{b}{1+b} \end{bmatrix}$$

1) At $(0,0)$: $J = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$ has trace > 0 and det < 0.

Has eigenvalues $\lambda_1 = \frac{1}{2} > 0$

$\lambda_2 = -\frac{1}{3} < 0$

Saddle point, unstable

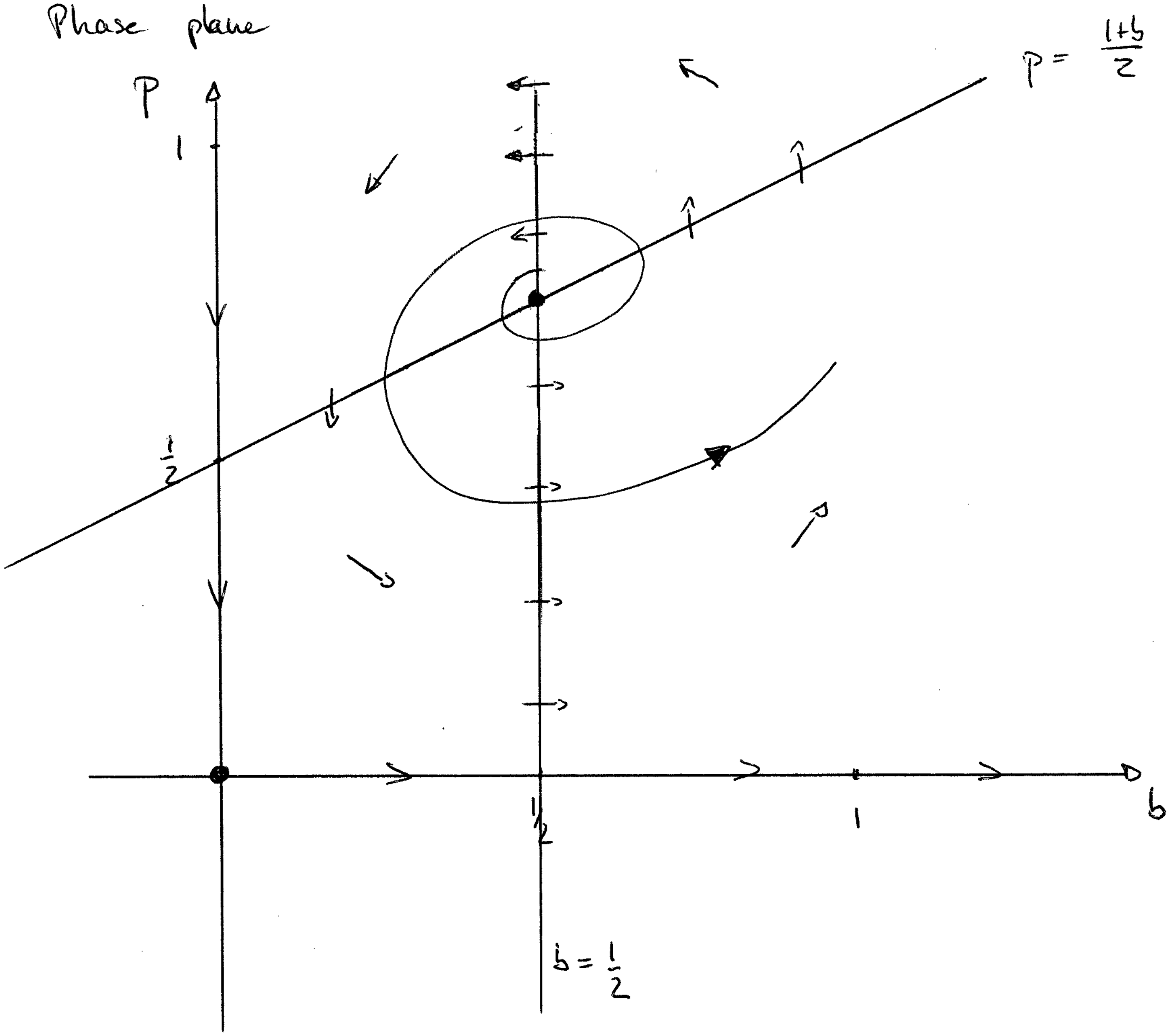
2) At $\left(\frac{1}{2}, \frac{3}{4}\right)$: $J = \begin{bmatrix} +\frac{1}{6} & -1/3 \\ \frac{1}{3} & 0 \end{bmatrix}$ has trace > 0 and det > 0

has eigenvalues $\lambda = \frac{1}{12} \pm \frac{\sqrt{15}}{12} i$

unstable, since $\text{Re}\,\lambda > 0$.

Question 1c, continued

5) Phase plane



$\frac{dp}{dt} > 0$ if $p > 0$ and $b > \frac{1}{2}$

$\frac{db}{dt} > 0$ if $b > 0$ and $p < \frac{1+b}{2}$