MAT 1332, Winter 2014, Assignment 2
Due Friday January 24 by 3:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ____________________________ Student Number ____________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________________

QUESTION 1. Find the area between the graphs of the functions $f(x) = x^{-2}$ and $g(x) = x^{-3}$ between $x = 1/2$ and $x = 2$.

- In order to compute the area between two curves we must first determine all the intersection points between the curves over the given interval $[1/2, 2]$.

$$\begin{aligned} f(x) &= g(x) \\ x^{-2} &= x^{-3} \\ \frac{1}{x^2} &= \frac{1}{x^3} \\ x &= 1 \text{ since } x \neq 0 \text{ by domain constraints.} \end{aligned}$$

- Next, we subdivide the given interval into subintervals separated by the intersection points determined above. Thus,

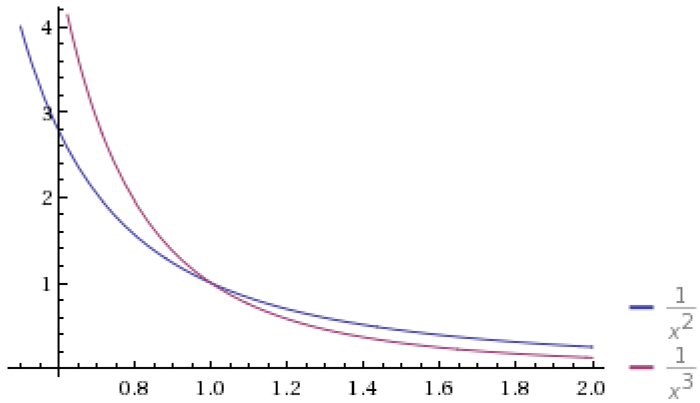
$$[1/2, 2] = [1/2, 1] \cup [1, 2].$$

- On each of the subintervals, we determine which of the two functions is greater. We can accomplish this either by looking at a graphical representation of the scenario or algebraically:

  Algebraically:

x	$[1/2, 1[$	$]1, 2]$
$f(x)$	$f(3/4) = 16/9 \approx 1,78$	$f(3/2) = \frac{4}{9} \approx 0.44$
$g(x)$	$g(3/4) = \frac{64}{27} \approx 2.37$	$g(3/2) = \frac{8}{27} \approx 0.30$

Graphically



In each case, we have determined that

$$g(x) \geq f(x) \qquad \text{on } [1/2, 1]$$

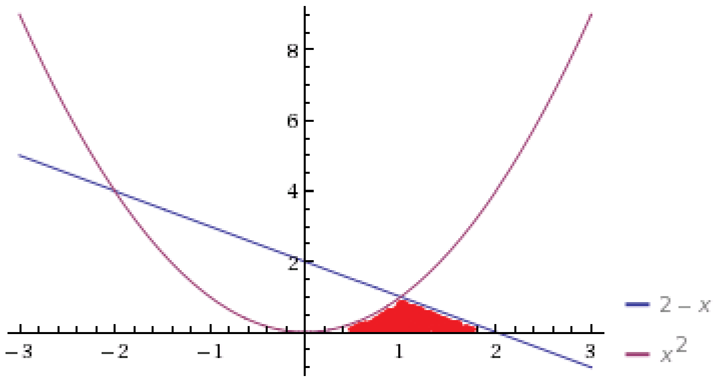
$$f(x) \geq g(x) \qquad \text{on } [1, 2]$$

- Now we can set up the integrals that compute the desired area:

$$\begin{aligned} A &= \int_{1/2}^{1} g(x) - f(x)\,dx + \int_{1}^{2} f(x) - g(x)\,dx \\ &= \int_{1/2}^{1} x^{-3} - x^{-2}\,dx + \int_{1}^{2} x^{-2} - x^{-3}\,dx \\ &= \left[-\frac{1}{2x^2} + \frac{1}{x}\right]_{1/2}^{1} + \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_{1}^{2} \\ &= \frac{1}{2} + \frac{1}{8} \\ &= \frac{5}{8} \end{aligned}$$

QUESTION 2. Consider the functions $f(x) = 2 - x$ and $g(x) = x^2$.

(a) Graph the two functions and shade the area bounded below by the x-axis and above by $\min\{f(x), g(x)\}$.



(b) Calculate the shaded area.

- By inspecting the shaded graph above, we determine that the area of the region is equal to the area under the x^2 curve from $[0, c]$ and the area under the $2 - x$ curve from $[c, 2]$, where c is the x-coordinate of the intersection between x^2 and $2 - x$.

$$\begin{aligned} x^2 &= 2 - x \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \end{aligned}$$

  Thus $c = 1$.

- Now we can set up the integrals that compute the desired area:

$$\begin{aligned} A &= \int_0^1 x^2\, dx + \int_1^2 2 - x\, dx \\ &= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 \\ &= \frac{1}{3} + \frac{1}{2} \\ &= \frac{5}{6} \end{aligned}$$

(c) Now assume that the shaded area rotates around the x-axis. Calculate the volume of the emerging shape.

- In general the area of a solid of revolution of this type is given by

$$\int_a^b \pi (R(x))^2 \, dx$$

where $R(x)$ represents the distance from the axis of rotation to the outer curve over the given interval.

- Since in our case, the outer curve changes over the given interval, we shall compute the volume over each subinterval $[0, 1]$ (where x^2 is the outer curve) and $[1, 2]$ (where $2 - x$ is the outer curve).

- Thus our volume computation is given as follows:

$$\begin{aligned} V &= \int_0^1 \pi (x^2)^2 \, dx + \int_1^2 \pi (2-x)^2 \, dx \\ &= \pi \int_0^1 x^4 \, dx + \pi \int_1^2 4 - 4x + x^2 \, dx \\ &= \pi \left[\frac{x^5}{5} \right]_0^1 + \pi \left[4x - 2x^2 + \frac{x^3}{3} \right]_1^2 \\ &= \frac{\pi}{5} + \frac{\pi}{3} \\ &= \frac{8\pi}{15} \end{aligned}$$

Question 3. Find the indefinite integral

$$\int \frac{x^3 + 1}{x^2 + 1} dx$$

- Since the degree of the numerator $deg(x^3 + 1) = 3$ is greater than the degree of the denominator $deg(x^2 + 1) = 2$, we need to use polynomial long division to rewrite this rational expression.

$$\begin{array}{r|l} & \quad x \\ \cline{2-2} x^2 + 1 & x^3 \qquad\qquad + 1 \\ & x^3 \quad + x \\ \cline{2-2} & \qquad - x + 1 \end{array}$$

- Thus
$$\frac{x^3+1}{x^2+1} = x + \frac{1-x}{x^2+1} = x + \frac{1}{x^2+1} - \frac{x}{x^2+1}.$$

- We now compute the indefinite integral
$$\begin{aligned}\int \frac{x^3+1}{x^2+1}\,dx &= \int x + \frac{1}{x^2+1} - \frac{x}{x^2+1}\,dx \\ &= \frac{x^2}{2} + \arctan x - \int \frac{x}{x^2+1}\,dx\end{aligned}$$

- Using substitution, with $u = x^2+1$ and $du = 2x\,dx$ we rewrite the remaining integral as
$$\begin{aligned}\int \frac{x}{x^2+1}\,dx &= \int \frac{x}{(u)}\left(\frac{du}{2x}\right) \\ &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln(x^2+1) + C\end{aligned}$$

- Therefore, putting everything together
$$\int \frac{x^3+1}{x^2+1}\,dx = \frac{x^2}{2} + \arctan x - \frac{1}{2}\ln(x^2+1) + C.$$

QUESTION 4. Find the indefinite integral
$$\int \frac{x^4+3}{x^2-4x+3}dx$$

- Since the degree of the numerator $deg(x^4+3) = 4$ is greater than the degree of the denominator $deg(x^2-4x+3) = 2$, we need to use polynomial long division to rewrite this rational expression.

	x^2	$+\ 4x$	$+\ 13$		
$x^2 - 4x + 3$	x^4				$+\ 3$
	x^4	$-\ 4x^3$	$+\ 3x^2$		
		$4x^3$	$-\ 3x^2$		$+\ 3$
		$4x^3$	$-\ 16x^2$	$+\ 12x$	
			$13x^2$	$-\ 12x$	$+\ 3$
			$13x^2$	$-\ 52x$	$+\ 39$
				$40x$	$-\ 36$

- Thus
$$\frac{x^4+3}{x^2-4x+13} = x^2+4x+13+\frac{40x-36}{x^2-4x+3}.$$

- In this case, $\frac{40x-36}{x^2-4x+3}$ is a rational function whose numerator's degree is strictly less than its denominator's degree.

- Since $x^2-4x+3=(x-3)(x-1)$ has two distinct roots, the method of partial fractions requires us to determine A and B in the following decomposition:
$$\begin{aligned} \frac{40x-36}{(x-3)(x-1)} &= \frac{A}{x-3}+\frac{B}{x-1} \\ 40x-36 &= A(x-1)+B(x-3) \\ 40x-36 &= (A+B)x-A-3B \end{aligned}$$

- By solving $A+B=40$ and $-A-3B=-36$, we determine that $A=42$ and $B=-2$.

- We now compute the indefinite integral
$$\begin{aligned} \int \frac{x^4+3}{x^2-4x+13}\,dx &= \int x^2+4x+13+\frac{40x-36}{x^2-4x+3}dx \\ &= \frac{x^3}{3}+2x^2+13x+\int \frac{40x-36}{x^2-4x+3}\,dx \\ &= \frac{x^3}{3}+2x^2+13x+\int \frac{42}{x-3}-\frac{2}{x-1}\,dx \\ &= \frac{x^3}{3}+2x^2+13x+42\ln|x-3|-2\ln|x-1|+C \end{aligned}$$