Solution to Assignment 4

MAT2322, Winter 2012

1. (2 marks) Let B be the solid bounded by the surfaces $y = x^2$, $y = \sqrt{x}$, $z = 0$, and $z = xy$. Find the triple integral $\iiint_B xyz\,dV$.

Solution. The base of the solid is the region D bounded by $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$, on the x-y plane (i.e., $z = 0$). Hence,

$$\iiint_B xyz\,dV = \iint_D \int_0^{xy} xyz\,dz\,dA = \int_0^1 x\left(\int_{x^2}^{\sqrt{x}} y\left(\int_0^{xy} z\,dz\right)dy\right)dx = \frac{1}{2}\int_0^1 x\int_{x^2}^{\sqrt{x}} x^2y^3\,dy\,dx$$

$$= \frac{1}{8}\int_0^1 x^3\left[y^4\right]_{y=x^2}^{\sqrt{x}}\,dx = \frac{1}{8}\int_0^1 x^3(x^2 - x^8)\,dx = \frac{1}{8}\left(\frac{1}{6} - \frac{1}{12}\right) = \frac{1}{96}.$$

2. (2 marks) Let B be the solid between the paraboloid $z = x^2 + y^2$ and the x-y plane, inside the cylinder $x^2 + y^2 = x$ in the first quadrant. Find the triple integral $\iiint_B \sqrt{x^2+y^2}\,dV$ by cylindrical coordinates.

Solution. The base of solid B is the upper half of the circle centered at $(1/2, 0)$ with radius $1/2$ in the x-y plane.

This region specified by cylindrical coordinates is $D = \{(r, \theta);\ 0 \le r \le \cos\theta,\ 0 \le \theta \le \pi/2\}$. Hence, letting $u = \sin\theta$, we have

$$\iiint_B \sqrt{x^2+y^2}\,dV = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r^2\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{\cos\theta} r^4\,dr\,d\theta = \frac{1}{5}\int_0^{\pi/2}\cos^5\theta\,d\theta$$

$$= \frac{1}{5}\int_0^1 (1-u^2)^2\,du = \frac{8}{75}.$$

3. (2 marks) Let C be the curve $\mathbf{r}(t) = (\sin^3 t, \cos^3 t)$, $0 \le t \le \pi/2$. Find the line integral $\int_C xy\,ds$.

You may need the following identity:

$$\sin^4\theta\cos^4\theta = \frac{1}{16}\sin^4(2\theta) = \frac{1}{64}(1-\cos(4\theta))^2 = \frac{1}{64}(1 - 2\cos(4\theta) + \cos^2(4\theta))$$

$$= \frac{1}{64}(1 - 2\cos(4\theta) + \frac{1}{2}(1+\cos(8\theta))) = \frac{1}{128}(3 - 4\cos(4\theta) + \cos(8\theta)).$$

This is obtained by $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta), \sin^2\theta = \frac{1}{2}(1-\cos(2\theta)), \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$.

Solution. $x(t) = \sin^3 t$, $x'(t) = 3\sin^2 t \cos t$. $y(t) = \cos^3 t$, $y'(t) = -3\cos^2 t \sin t$.

$(x'(t))^2 + (y'(t))^2 = 9(\sin^4 t \cos^2 t + \cos^4 t \sin^2 t) = 9\sin^2 t \cos^2 t$.

$\sqrt{(x'(t))^2 + (y'(t))^2} = 3\sin t \cos t$.

$\int_C xyds = \int_0^{\pi/2} \sin^3 t \cos^3 t(3\sin t \cos t)dt = 3\int_0^{\pi/2} \sin^4 t \cos^4 tdt = \frac{9}{256}\pi$.

4. (2 marks) Consider curve C: $\mathbf{r}(t) = (e^t \cos t, e^t \sin t, e^t)$, $0 \le t \le 1$, and vector field $\mathbf{F}(x, y, z) = (x, y, z)$. Find line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Solution. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (e^t \cos t)d(e^t \cos^t) + (e^t \sin t)d(e^t \sin t) + e^t de^t = \int_0^1 2e^{2t}dt = e^2 - 1$.

5. (2 marks) Consider the vector field $\mathbf{F}(x, y) = (2xy + \cos x, x^2 - \sin y)$.

(a) Show that this vector field is conservative.

(b) Find the general potential function of this field.

Solution. (a) Since $\frac{\partial}{\partial y}(2xy + \cos x) = 2x = \frac{\partial}{\partial x}(x^2 - \sin y)$, this field is conservative.

(b) $f(x, y) = \int (2xy + \cos x)dx = x^2 y + \sin x + g(y)$.

$\frac{\partial f(x, y)}{\partial y} = x^2 + g'(y) = x^2 - \sin y, g'(y) = -\sin y, g(y) = \cos y + C$.

The general potential function of this vector field is $f(x, y) = x^2 y + \sin x + \cos y + C$.