Solution to Assignment 2

MAT 2322, Winter 2012
2 × 5 = 10 marks

1. (2 marks) Consider curve $\mathbf{r}(t) = \left(t,\ t^2,\ \frac{2}{3}t^3\right)$.

(a) Find the length of the arc $0 \le t \le 1$.

(b) Find an equation of the tangent line of the curve at $t = 3$.

Solution. (a) $\mathbf{r}'(t) = (1, 2t, 2t^2)$. $|\,\mathbf{r}'(t)\,| = \sqrt{1+4t^2+4t^4} = 1+2t^2$.

$L = \int_0^1 (1+2t^2)dt = \frac{5}{3}$.

(b) The tangent line has a direction vector $\mathbf{r}'(3) = (1, 6, 18)$.

$\mathbf{r}(3) = (3, 9, 18)$. An equation of the tangent line at the point $t = 3$ is

$(x, y, z) = (3, 9, 18) + (1, 6, 18)\, t$.

2. (2 marks) Consider space curve $\mathbf{r}(t) = \left(t, \frac{2\sqrt{2}}{3}t^{3/2}, \frac{1}{2}t^2\right)$. Find the unit tangent vector $\mathbf{T}(t)$, the principle normal vector $\mathbf{N}(t)$, the binormal vector $\mathbf{B}(t)$, the curvature $\kappa(t)$, and the radius of curvature $R(t)$, as functions of t.

Solution. $\mathbf{r}'(t) = (1, \sqrt{2t}, t)$, $|\,\mathbf{r}'(t)\,| = \sqrt{1+2t+t^2} = 1 + t$.

$\mathbf{T}(t) = \frac{1}{1+t}(1, \sqrt{2t}, t)$. $\mathbf{T}'(t) = \frac{1}{(1+t)^2}\left(-1, \frac{1-t}{\sqrt{2t}}, 1\right)$.

$|\,\mathbf{T}'(t)\,| = \frac{1}{(1+t)^2}\sqrt{1+\frac{(1-t)^2}{2t}+1} = \frac{1}{\sqrt{2t}(1+t)}$.

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{1+t}\left(-\sqrt{2t}, 1-t, \sqrt{2t}\right)$.

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{(1+t)^3}\left(2t-t+t^2, -t\sqrt{2t}-\sqrt{2t}, 1-t+2t\right) = \frac{1}{(1+t)^2}\left(t, -\sqrt{2t}, 1\right)$.

$\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2t}(1+t)^2}$, $R(t) = (1+t)^2\sqrt{2t}$.

3. (2 marks) Consider vector function $\mathbf{r}(t) = (2t, e^{-t}, 2e^t)$, $-\infty < t < \infty$.

(a) (0.5 mark) Find $\mathbf{r}'(t)$ and the length of $\mathbf{r}'(t)$.
(b) (1 mark) Find the curvature and radius of curvature of the curve at the point where $t = 0$.
(c) (0.5 mark) Find an equation of the osculating plane of the curve at the point $t = 0$.

Solution. (a) $\mathbf{r}(0) = (0, 1, 2)$.

$\mathbf{r}'(t) = (2, -e^{-t}, 2e^t)$. $|\mathbf{r}'(t)| = \sqrt{4 + e^{-2t} + 4e^{2t}} = e^{-t} + 2e^t = \dfrac{1+2e^{2t}}{e^t}$.

(b) $\mathbf{T}(t) = \dfrac{1}{1+2e^{2t}}(2e^t, -1, 2e^{2t})$. $\mathbf{T}'(t) = \dfrac{2e^t}{(1+2e^{2t})^2}\left(1-2e^{2t}, 2e^t, 2e^t\right)$.

At the point $t = 1$, $\mathbf{T}'(0) = \dfrac{2}{9}(1, 2, 2)$. $|\mathbf{T}'(0)| = \dfrac{2}{3}$. $|\mathbf{r}'(0)| = 3$. The curvature is $\dfrac{2}{9}$.

The radius of curvature is $\dfrac{9}{2}$.

(c) The tangent line is in the direction of $\mathbf{t} = \mathbf{r}'(0) = (2, -1, 2)$. The principal normal vector $\mathbf{N}$ is in the direction of $\mathbf{T}'(0)$, which is in the direction of vector $\mathbf{n} = (-1, 2, 2)$.

A normal vector of the osculating plane is $\mathbf{t} \times \mathbf{n} = (-6, -6, 3)$.

The osculating plane is $(-6, -6, 3) \cdot (x, y - 1, z - 2) = 0$, or $2x + 2y - z = 0$.

Recall that this equation may be multiplied by any non-zero number. Hence, the following are also correct answers:

$4x + 4y - 2z = 0$, or $\dfrac{2}{3}x + \dfrac{2}{3}y - \dfrac{1}{3}z = 0$.

4. (2 marks) Find all critical points of the function $z = 3x - x^3 - 2y^2 + y^4$, and use the second derivative test to classify each critical point as a local minimum, or a local maximum, or a saddle point.

Solution. $z_x = 3 - 3x^2 = 3(1 - x^2)$. $z_y = -4y + 4y^3 = -4y(1 - y^2)$ There are six critical points: (1, 1), (1, 0), (1, –1), (–1, 1), (–1, 0) and (–1, –1).

$z_{xx} = -6x$, $z_{yy} = -4 + 12y^2$, $z_{xy} = 0$. $D = z_{xx}z_{yy} - z_{xy}^2 = 24x - 72xy^2 = 24x(1 - 3y^2)$.

When $(x, y) = (1, 1), (1, -1)$, and $(-1, 0)$, $D < 0$. These are saddle points.

When $(x, y) = (1, 0), (-1, 1)$, and $(-1, -1)$, $D > 0$.

$(1, 0)$ is a local minimum, and $(-1, 1)$ and $(-1, -1)$ are local maxima.

5. (2 marks) Use Lagrange multiplier to find the maximum and minimum value of the function $z = 2x + y$ with constraint $x^2 + y^2 = 1$.

Solution. $F(x, y, \lambda) = (2x + y) + \lambda(x^2 + y^2 - 1)$.

$F_x = 2 + 2x\lambda = 0$, $F_y = 1 + 2y\lambda = 0$, $F_\lambda = x^2 + y^2 - 1 = 0$.

From the first two equations, $\lambda = -1 / x = -1 / (2y)$. $x = 2y$. Hence, $5y^2 = 1$, $y = \pm\frac{1}{\sqrt{5}}, x = \pm\frac{2}{\sqrt{5}}$.

There are two critical points: $\mathbf{p}_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \mathbf{p}_2 = \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$. At $\mathbf{p}_1$, $z_{\text{max}} = \sqrt{5}$, and at $\mathbf{p}_2$, $z_{\text{min}} = -\sqrt{5}$.