Solution to Assignment 3
MAT2322
Winter 2012

1. Find the double integral $\iint_D \sqrt{x^2+3}\,dA$, where D is the region in the first quadrant bounded by lines $y = 2x$ and $x = 0$, and the graph of the equation $y^2 - x^2 = 3$.

Solution. The intersection of $y = 2x$, and $y^2 - x^2 = 3$ is obtained by $(2x)^2 - x^2 = 3$, $x^2 = 1$, $x = 1$. Then $y = 2$.

$$\iint_D \sqrt{x^2+3}\,dA = \int_0^1\int_{2x}^{\sqrt{x^2+3}} \sqrt{x^2+3}\,dydx = \int_0^1 \sqrt{x^2+3}(\sqrt{x^2+3}-2x)dx$$

$$= \int_0^1 (x^2+3-2x\sqrt{x^2+3})dx = \left[\frac{x^3}{3}+3x-\frac{2}{3}(x^2+3)^{3/2}\right]_{x=0}^1 = \frac{1}{3}+3-\frac{16}{3}+2\sqrt{3} = 2(\sqrt{3}-1).$$

2. Find the iterated integral $\int_0^1\int_y^1 e^{x^2}\,dxdy$ by reversing the order of integration.

Solution. $\int_0^1\int_y^1 e^{x^2}\,dxdy = \iint_D e^{x^2}\,dA$, where D is the triangle enclosed by lines $y = 0$, $y = x$, and $x = 1$. Regard region D as a type I region, this integral is

$$\int_0^1\int_0^x e^{x^2}\,dydx = \int_0^1 xe^{x^2}\,dx = \frac{1}{2}\left[e^{x^2}\right]_{x=0}^1 = \frac{1}{2}(e-1).$$

3. Find the double integral $\iint_D xy\,dA$ using polar coordinates, where region D is the region under the line $y = x$, above the x-axis, and in the circle $(x - 1)^2 + y^2 = 1$.

Solution.

$$\iint_D xy\,dA = \int_0^{\pi/4}\int_0^{2\cos\theta} r^3\sin\theta\cos\theta\,drd\theta = \frac{1}{4}\int_0^{\pi/4}\sin\theta\cos\theta\left[r^4\right]_{r=0}^{2\cos\theta} = 4\int_0^{\pi/4}\sin\theta\cos^5\theta\,d\theta$$

$$= -\frac{2}{3}\left[\cos^6\theta\right]_{\theta=0}^{\pi/4} = -\frac{2}{3}\left(\frac{1}{8}-1\right) = \frac{7}{12}.$$

4. Find the area of the parametric surface

$\mathbf{r}(\theta, \alpha) = (x, y, z) = (\cos\theta(2 + \cos\alpha), \sin\theta(2 + \cos\alpha), \sin\alpha)$, $0 \le \theta \le 2\pi$, $0 \le \alpha \le 2\pi$.

(See question 22, page 872, in your textbook).

Solution. $\mathbf{r}_\theta = (-\sin\theta(2 + \cos\alpha), \cos\theta(2 + \cos\alpha), 0)$,

$\mathbf{r}_\alpha = (-\cos\theta\sin\alpha, -\sin\theta\sin\alpha, \cos\alpha)$.

$\mathbf{r}_\theta \times \mathbf{r}_\alpha = (\cos\theta\cos\alpha(2 + \cos\alpha), \sin\theta\cos\alpha(2 + \cos\alpha), \sin^2\theta\sin\alpha(2 + \cos\alpha)$
$+ \cos^2\theta\sin\alpha(2 + \cos\alpha)) = (2 + \cos\alpha)(\cos\theta\cos\alpha, \sin\theta\cos\alpha, \sin\alpha)$.

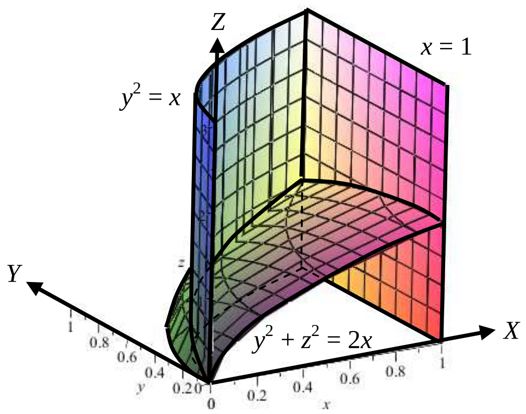
$|\mathbf{r}_\theta \times \mathbf{r}_\alpha| = 2 + \cos\alpha$.

The area of the surface is

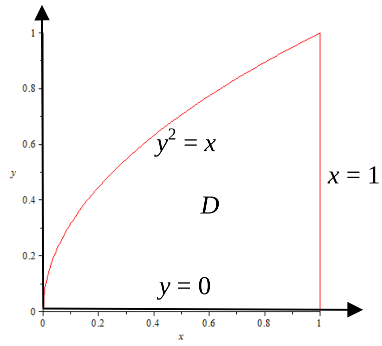
$$\int_0^{2\pi}\int_0^{2\pi}(2+\cos\alpha)\,d\theta\,d\alpha = 2\pi\left[2\alpha - \sin\alpha\right]_{\alpha=0}^{2\pi} = 8\pi^2.$$

5. Find the area of the surface S, which is the part of the graph of the function $y^2 + z^2 = 2x$ inside the cylinder bounded by surfaces $y^2 = x$, and two planes $x = 1$ and $y = 0$, in the first octant.
(Note that you may need the formula $\int \frac{1}{\sqrt{a^2 - y^2}}\,dy = \arcsin\frac{y}{a}$).

Solution. This surface looks like the following:



The projection of the surface onto x-y plane is shown as the following figure:



$z = \sqrt{2x - y^2}$, $z_x = \dfrac{1}{\sqrt{2x-y^2}}$, $z_y = -\dfrac{y}{\sqrt{2x-y^2}}$.

$$\sqrt{1+z_x^{\ 2}+z_y^{\ 2}} = \sqrt{1+\frac{1}{2x-y^2}+\frac{y^2}{2x-y^2}} = \sqrt{\frac{2x+1}{2x-y^2}}.$$

Hence, the area of the surface is

$A = \iint_D \sqrt{\dfrac{2x+1}{2x-y^2}}\,dA$, where D is the region on the x-y plane bounded by the lines $x = 1$, $y = 0$, and the graph of $y = \sqrt{x}$ as shown in the figure above.

By Fubini's Theorem, we have

$$A = \int_0^1\int_0^{\sqrt{x}} \frac{\sqrt{2x+1}}{\sqrt{2x-y^2}}\,dydx = \int_0^1 \sqrt{2x+1}\left(\int_0^{\sqrt{x}} \frac{1}{\sqrt{2x-y^2}}\,dy\right)dx$$

Use a variable substitution, $u = \dfrac{y}{\sqrt{2x}}$, $\displaystyle\int_0^{\sqrt{x}} \frac{1}{\sqrt{2x-y^2}}\,dy = \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-u^2}}\,du = \left[\arcsin u\right]_{u=0}^{1/\sqrt{2}} = \frac{\pi}{4}$.

Therefore, $A = \dfrac{\pi}{4}\displaystyle\int_0^1 \sqrt{2x+1}\,dx = \frac{\pi}{8}\left[\frac{2}{3}(2x+1)^{3/2}\right]_{x=0}^{1} = \frac{\pi}{12}\left(3\sqrt{3}-1\right)$.

6. Find the are of the surface obtained by revolving the curve $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$, $1 \le x \le 2$, about the x-axis. (Recall that $x^4 + \dfrac{1}{2} + \dfrac{1}{16x^4} = \left(x^2 + \dfrac{1}{4x^2}\right)^2$).

Solution. $y' = x^2 - \dfrac{1}{4x^2}$, $\sqrt{1+(y')^2} = \sqrt{1+x^4-\dfrac{1}{2}+\dfrac{1}{16x^4}} = \sqrt{x^4+\dfrac{1}{2}+\dfrac{1}{16x^4}} = x^2+\dfrac{1}{4x^2} = \dfrac{4x^4+1}{4x^4}$.

The area is

$$A = 2\pi\int_1^2\left(\frac{x^3}{3}+\frac{1}{4x}\right)\left(\frac{4x^4+1}{4x^2}\right)dx = 2\pi\int_1^2\left(\frac{4x^4+3}{12x}\right)\left(\frac{4x^4+1}{4x^2}\right)dx = \frac{\pi}{24}\int_0^1\frac{16x^8+16x^4+3}{x^3}\,dx$$

$$= \frac{\pi}{24}\int_0^1\frac{16x^8+16x^4+3}{x^3}\,dx = \frac{\pi}{24}\left[\frac{8}{3}x^6+8x^2-\frac{3}{2x^2}\right]_{x=1}^{2} = \frac{515}{64}\pi.$$