

Solution to Assignment 1

MAT2322, Winter 2012

1. (2 marks) Consider vectors $\mathbf{u} = (2, 4, 4)$ and $\mathbf{v} = (3, -6, 2)$.

(a) Find $\mathbf{u} \cdot \mathbf{v}$.

(b) Find the angle between \mathbf{u} and \mathbf{v} .

(c) Find the projection of \mathbf{u} onto \mathbf{v} , $\text{proj}_{\mathbf{v}} \mathbf{u}$.

(d) Find the cross product $\mathbf{u} \times \mathbf{v}$.

Solution. (a) $\mathbf{u} \cdot \mathbf{v} = 6 - 24 + 8 = -10$.

(b) $|\mathbf{u}| = 6$, $|\mathbf{v}| = 7$. $\cos \theta = -10 / (6 \times 7) = -0.2381$, $\theta = 1.8112$.

(c) $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = -\frac{10}{49} \mathbf{v} = \left(-\frac{30}{49}, \frac{60}{49}, -\frac{20}{49} \right)$.

(d) $\mathbf{u} \times \mathbf{v} = (32, 8, -24)$.

2. (2 marks) Find the equation of a plane in the general form $ax + by + cz = d$ that contains a line L given by the vector equation $(x, y, z) = (1, 2, -1) + (-4, 1, 0)t$, and a point $\mathbf{p} = (2, -1, 2)$.

Solution. $\mathbf{q} = (1, 2, -1)$ is a point on L . Vector $\mathbf{p} - \mathbf{q} = (1, -3, 3)$ is in the plane. The direction vector $\mathbf{v} = (-4, 1, 0)$ is in the plane. Hence, $\mathbf{n} = (\mathbf{p} - \mathbf{q}) \times \mathbf{v} = (-3, -12, -11)$ is normal to this plane. The vector form of the plane is

$$(-3, -12, -11) \cdot ((x, y, z) - (1, 2, -1)) = 0.$$

The general form of the equation of the plane is $3x + 12y + 11z = 16$.

3. (2 marks) Find the distance d from a point $\mathbf{p} = (2, 5, 0)$ to the line $L: \frac{x-1}{2} = -\frac{y}{2} = z+1$.

Solution 1. $\mathbf{r} = \mathbf{p} - \mathbf{q} = (1, 5, 1)$. The projection of \mathbf{r} onto \mathbf{v} is

$$\text{proj}_{\mathbf{v}} \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{-7}{9} (2, -2, 1) = \left(-\frac{14}{9}, \frac{14}{9}, -\frac{7}{9} \right).$$

$\mathbf{u} = \mathbf{r} - \text{proj}_{\mathbf{v}} \mathbf{r} = \left(\frac{23}{9}, \frac{31}{9}, \frac{16}{9} \right)$. The distance from \mathbf{p} to this line is

$$|\mathbf{u}| = \frac{1}{9} \sqrt{23^2 + 31^2 + 16^2} = \frac{\sqrt{1746}}{9} = \frac{\sqrt{194}}{3}.$$

Solution 2. The direction vector of L is $\mathbf{v} = (2, -2, 1)$. Take a point $\mathbf{q} = (1, 0, -1)$ on L . Then $\mathbf{p} - \mathbf{q} = (1, 5, 1)$.

$$\mathbf{v} \times (\mathbf{p} - \mathbf{q}) = (-7, -1, 12). \quad d = |\mathbf{v} \times (\mathbf{p} - \mathbf{q})| / |\mathbf{v}| = \frac{\sqrt{194}}{3}.$$

4. (2 marks) Find the distance from point $\mathbf{p} = (3, -2, 5)$ to plane $2x - 6y + 3z = -1$.

Solution 1. Take a point $\mathbf{q} = (1, 1, 1)$ on the plane. $\mathbf{r} = \mathbf{p} - \mathbf{q} = (2, -3, 4)$. A normal vector of the plane is $\mathbf{n} = (2, -6, 3)$. The distance from \mathbf{p} to the plane is the length of the projection of \mathbf{r} to \mathbf{n} :

$$d = \frac{|\mathbf{r} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|4 + 18 + 12|}{\sqrt{4 + 36 + 9}} = \frac{34}{7}.$$

Solution 2. Take three points $\mathbf{r} = (1, 1, 1)$, $\mathbf{s} = (1, 0, -1)$ and $\mathbf{t} = (-5, -1, 1)$ on the plane. $\mathbf{a} = \mathbf{s} - \mathbf{r} = (0, -1, 2)$, $\mathbf{b} = \mathbf{t} - \mathbf{r} = (-6, -2, 0)$, $\mathbf{c} = \mathbf{p} - \mathbf{r} = (2, -3, 4)$. Then the distance from \mathbf{p} to the plane is

$$d = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|} = \frac{|(4, -12, 6) \cdot (2, -3, 4)|}{|(4, -12, 6)|} = \frac{|8 + 36 + 24|}{\sqrt{16 + 144 + 36}} = \frac{34}{7}.$$

5. (2 marks) Show the following lines do not intersect, and find the distance between these two lines:

$$L_1: x = 2s, y = 2s - 1, z = s + 1,$$

$$L_2: x = t, y = 2t - 2, z = 3.$$

Solution. Let $2s = t$, $2s - 1 = 2t - 2$, $s + 1 = 3$. We have $s = 2$. From the first equation, $t = 4$. However, $t = 4$ and $s = 2$ is not a solution to the second equation. Hence, these two lines do not intersect.

$\mathbf{v}_1 = (2, 2, 1)$ is a direction vector of L_1 , $\mathbf{v}_2 = (1, 2, 0)$ is a direction vector of L_2 .

A vector perpendicular to both lines is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-2, 1, 2)$.

A vector from $\mathbf{q} = (0, -1, 1)$ on L_1 to $\mathbf{p} = (0, -2, 3)$ on L_2 is $\mathbf{u} = \mathbf{p} - \mathbf{q} = (0, -1, 2)$. The component of \mathbf{u} onto \mathbf{n} is the distance $d = \frac{|\mathbf{un}|}{|\mathbf{n}|} = \frac{3}{3} = 1$.