Faculty of Science »
Department of mathematics and statistics

MAT 1341A Fall 2014 Final Exam

13 December, 2014.

Instructor - Barry Jessup

Family Name:____________________

First Name:____________________ Seat number: ________

Student number:____________________

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Some Advice

Take a few minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and questions 11-15 are worth 6 points each. Make a note of the questions you feel confident you can do, and try those first: you do not have to do the questions in the order they are presented.

Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. **The use of calculators, cell phones, or similar devices is not permitted.** All implanted cyber devices not necessary for life-support must be disabled at the beginning of the exam.
3. Questions 1 to 10 are multiple choice. These questions have just one correct answer, are worth 1 point each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and are worth 6 points each.
   Question 16 is a bonus question and should only be attempted after all other questions have been completed and checked.

   Spend your time accordingly.

   Answer questions 11 – 16 in the space provided, and use the backs of pages if necessary.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

Good luck! Bonne chance!

1. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x \geq 0,\ y \geq 0 \text{ and } z \geq 0\}$. Which one of the following statements is true?

A. $(0, 0, 0) \in X$ and X is closed under multiplication by scalars.

B. $(0, 0, 0) \notin X$ but X is closed under addition.

C. X is closed under addition but X is not closed under multiplication by scalars.

D. X is closed under addition and X is closed under multiplication by scalars.

E. X is not closed under addition but X is closed under multiplication by scalars.

F. None of the other statements is true.

2. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent if $a = b = c = 0$ implies $au + bv + cw = 0$.

II. A set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ implies $a = b = c = 0$.

III. A set $\{u, v, w\} \subset V$ of vectors spans a vector space V if every vector in V is a linear combination of u and w.

IV. $\{(1, -1), (1, 1)\}$ is an orthogonal set in $\mathbf{R}^2$.

A. Only I & II

B. Only II & IV

C. Only II & III

D. Only I & III & IV

E. Only II & III & IV

F. All of the above statements are true.

3. In a linear system $Ax = b$, with n equations and n unknowns, the rank of A is $n-1$ and the rank of the augmented matrix $[A \mid b]$ is also $n-1$. Which one of the following statements is true?

A. The system has no solution.

B. The system has a unique solution.

C. The system has infinitely many solutions.

D. The system has exactly $n-1$ solutions.

E. The determinant of A is non-zero.

F. Such a system cannot exist.

4. If C is an $m \times 2$ matrix and $D = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$, then the second column of the matrix CD is

A. not defined unless $m = 2$.

B. twice the first column of C.

C. the same as the first column of C.

D. the same as the second column of C.

E. the sum of the first and the second columns of C.

F. the sum of twice the first column of C and three times the second column of C.

5. The dimension of $S = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid A = A^t\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

6. Let A be a 7×12 matrix such that $\operatorname{rank} A = 6$. As usual, $\ker A = \{x \in \mathbf{R}^{12} \mid Ax = 0\}$ and col $A = \{Ax \mid x \in \mathbf{R}^{12}\}$. Which of the following statements is true?

A. $\dim \operatorname{col} A = 6, \dim \ker A = 1$

B. col $A = \mathbf{R}^{12}, \dim \ker A = 6$

C. col $A = \mathbf{R}^{7}, \dim \ker A = 5$

D. $\dim \operatorname{col} A = 5, \ker A = \mathbf{R}^{12}$

E. $\dim \operatorname{col} A = 6, \dim \ker A = 6$

F. col $A = \{0\}, \dim \ker A = 5$

7. If two $n \times n$ matrices A and B satisfy $A^t = B^{-1}$ and $B^t = -B^{-1}$ then $(ABA)^t$ is always

A. $-B^3$
B. B^2A
C. $-B^{-3}$
D. B^{-3}
E. B^3
F. AB^2

8. Which two of the following statements are **false**?

(i) For all invertible $n \times n$ matrices A and B, $\det(A^{-1}BA) = \det B$

(ii) For all invertible $n \times n$ matrices A and B, $\det(A^{-1}B^{-1}AB) = 1$

(iii) For all $n \times n$ matrices A and B, $(A^tB^t)^t = AB$

(iv) For all invertible $n \times n$ matrices A and B, $(ABA^{-1})^{-1} = A^{-1}B^{-1}A$

(v) For all $n \times n$ matrices A and B, $\det(A^tB) = \det(B^tA)$

A. (i) and (iii)

B. (ii) and (iii)

C. (iii) and (iv)

D. (ii) and (iv)

E. (ii) and (v)

F. (i) and (v)

9 . If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix}$.

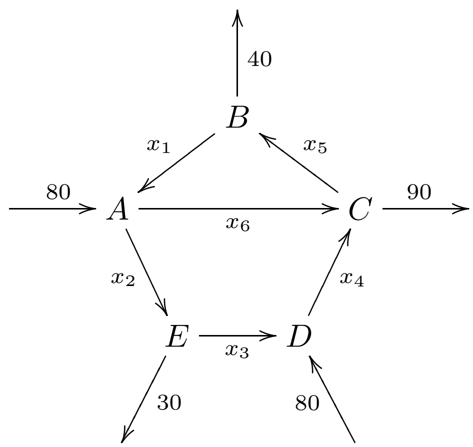
A. 24
B. -24
C. 12
D. -12
E. 6
F. -6

10. Which of the following are linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{\cos x, \sin x\}$
$T = \{1, \cos^2 x, \sin^2 x\}$
$U = \{1, 2\cos^2 x, 3\sin^2 x\}$
$V = \{1, \cos x, \sin x\}$

A. T and V.
B. T and U.
C. S and T.
D. S and V.
E. S, U and V.
F. S, U and T.

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables x_i, $i = 1, \ldots, 6$.

(**Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.**)

11(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

c) If $\overline{AC}$ were closed due to roadwork, find the minimum flow along $\overline{ED}$, **using your results from (b).**

(You must justify all your answers.)

12. Let $v_1 = (1, 0, 0, -1)$, $v_2 = (1, -1, 0, 0)$, $v_3 = (1, 0, 1, 0)$ and set $U = \text{span}\{v_1, v_2, v_3\} \subset \mathbf{R}^4$.

a) Briefly explain (you may refer to results learned in class or from the book) why U is a subspace of $\mathbf{R}^4$, and why $\{v_1, v_2, v_3\}$ is a basis for U.

b) Use the **Gram-Schmidt algorithm** to find an orthogonal basis for U.

c) Find the best approximation by a vector in U to the vector $(2, 0, 2, 4)$.

13. Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$.

a) Find the characteristic polynomial of A, and use this to show that the eigenvalues of A are 5 and -1.

b) Find a basis of $E_5 = \{v \in \mathbf{R}^3 \mid Av = 5v\}$.

c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If this is not possible, explain why.

14. Let $u = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(v) = u \times v, \qquad \text{for all } v \in \mathbf{R}^3,$$

where "$u \times v$" denotes cross product of u and v. *(You do <u>not</u> have to prove that S is linear.)*

a) If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3$, show that $S(v) = \begin{bmatrix} -2y + z \\ 2x - 2z \\ -x + 2y \end{bmatrix}$.

b) Find a 3×3 matrix A such that $S(v) = Av$, where Av denotes the matrix product of A and v.

c) Find a basis for $\ker S = \{v \mid S(v) = 0\}$, and give a complete geometric description of $\ker S$.

d) Find a basis for $\text{im } S = \{S(v) \mid v \in \mathbf{R}^3\}$, and give a complete geometric description of $\text{im } S$.

15. a) State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

i) Suppose C is an invertible 3×3 matrix, and that $\{v_1, v_2, v_3\}$ is linearly independent in $\mathbf{R}^3$. Then $\{Cv_1, Cv_2, Cv_3\}$ is also linearly independent.

ANSWER

ii) $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable.

ANSWER

15 a) (cont.)

iii) If the columns of a 17×6 matrix A form an orthogonal set in $\mathbf{R}^{17}$, then $\operatorname{rank} A = 6$.

ANSWER

15 b). Let A be a $n \times n$ matrix with real entries. Give three additional statements equivalent to

" A is **not** invertible"

one each in terms of

(i) the homogeneous linear system $Ax = 0$:

(ii) the rank of A:

(iii) the determinant of A:

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

In parts (a) and (b), A denotes a symmetric 100×100 matrix.

a) Prove that $(Au) \cdot u' = u \cdot (Au')$ for all $u, u' \in \mathbf{R}^{100}$, where "$\cdot$" denotes the dot product.

Now let $v_\lambda \in \mathbf{R}^{100}$ be an eigenvector of A with eigenvalue λ, and set $W = \{w \in \mathbf{R}^{100} \mid w \cdot v_\lambda = 0\}$.

b) Prove that if $w \in W$, then $Aw \in W$.