Honors Linear Algebra-Homework 2

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1 1.3 Subspaces

1.Label the following statements as true or false.

(a)If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

(b)The empty set is a subspace of every vector space.

(c)If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(d)The intersection of any two subsets of V is a subspace of V.

(e)An $n \times n$ diagonal matrix can never have more than more than n nonzero entries.

(f)The trace of a square matrix is the product of its diagonal entries.

(g)Let W be the xy-plane in $\mathbb{R}^3$; that is, $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$.

(a) is false because the operations might be different.

(b) is false because there is no zero element.

(c) is true because we can take W to be the zero subspace.

(d) is false. Trivial.

(e) is true.

(f) is false. Trace is the sum of its diagonal entries.

(g) is false. W and $\mathbb{R}^2$ are different set.

8.Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(a)$W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

(b)$W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

(f)$W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

To check a subset of a vector space V is a subspace, we need to check the following three conditions: zero element is in the subset, the subset is closed under addition, the subset is closed under scalar multiplication.

(a) is a subspace. Very easy to check.

(b) is not. Because the zero element $(0,0,0)$ of $\mathbb{R}^3$ is not in W_2.

(f) is not. Because W_6 is not closed under addition. We take $(3,\sqrt{15},0),(3,-\sqrt{15},0)\in W_6$. $(3,\sqrt{15},0)+(3,-\sqrt{15},0)=(6,0,0)$ is not in W_6.

11.Is the set $W=\{f(x)\in P(F): f(x)=0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $P(F)$ if $n\neq 1$? Justify your answer.

No. $x^n, x^n+x^{n-1}\in W$, but $x^n-(x^n+x^{n-1})=-x^{n-1}\notin W$. Not closed under addition.

13.Let S be a nonempty set and F is a field. Prove that for any $s_0\in S$, $\{f\in\mathcal{F}(S,F): f(s_0)=0\}$ is a subspace of $\mathcal{F}(S,F)$.

Proof.Denote $V=\{f\in\mathcal{F}(S,F): f(s_0)=0\}$. Then the zero element of $\mathcal{F}(S,F)$, $0_S: S\to F, s\mapsto 0$, is in V. Now let $f,g\in V$, $c\in F$, that means $f(s_0)=g(s_0)=0$. So $(f+g)(s_0)=f(s_0)+g(s_0)=0$, $(cf)(s_0)=cf(s_0)=0$, hence V is closed under addition and scalar multiplication. So V is a subspace of $\mathcal{F}(S,F)$.

17.Prove that a subset W of a vector space V is a subspace of V if and only if $W\neq\varnothing$, and whenever $a\in F$ and $x,y\in W$, then $ax\in W$ and $x+y\in W$.

Proof." $\Rightarrow$ " The only thing need to show is $W\neq\varnothing$. This is easy since $0\in W$.

" $\Leftarrow$ " The only thing need to show is $0\in W$. Since $W\neq\varnothing$, we take an element s from W, then $-1\cdot s\in W$, and $0=s+(-s)\in W$.

19.Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1\bigcup W_2$ is a subspace of V if and only if $W_1\subset W_2$ or $W_2\subset W_1$.

Proof. " $\Leftarrow$ " is trivial. Because $W_1\cap W_2=W_1$ or W_2, in both cases, $W_1\cap W_2$ is a subspace.

" $\Rightarrow$ " We prove this by contradiction. Suppose $W_1\not\subseteq W_2$ and $W_2\not\subseteq W_1$, then there exist $s_1\in W_1$, $s_2\in W_2$, but $s_1\notin W_2$, $s_2\notin W_1$. Now $s_1,s_2\in W_1\bigcup W_2$. Because $W_1\bigcup W_2$ is a subspace, $s_1+s_2\in W_1\bigcup W_2$. Suppose $s_1+s_2\in W_1$(the case when $s_1+s_2\in W_2$ is very similar), then $s_2=(s_1+s_2)-s_1\in W_1$. Contradiction.

21.Show that the set of convergent sequences $\{a_n\}$ us a subspace of the vector space V in Exercise 20 of Section 1.2.

Proof. We denote the set of convergent sequences by W. The zero element of the sequence space, $(0,0,\cdots)$, is convergent, hence in W. If $a=(a_1,a_2\cdots,),b=(b_1,b_2,\cdots)\in W$, and $c\in\mathbb{R}$, then $a+b$ and ca are still convergent. Hence in W. So W is a subspace of V.

23.Let W_1 and W_2 be subspaces of a vector space V.

(a)Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b)Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Proof. (a)Check the three conditions for subspace. $0 = 0 + 0 \in W_1 + W_2$. If $a_1 + a_2, b_1 + b_2 \in W_1 + W_2$, $c \in F$, then $a_1 + a_2 + b_1 + b_2 = (a_1 + b_1) + (a_2 + b_2) \in W_1 + W_2$, and $c(a_1 + a_2) = ca_1 + ca_2 \in W_1 + W_2$.

(b)If W is a subspace of V containing W_1 and W_2. Let $a_1 + a_2 \in W_1 + W_2$ with $a_1 \in W_1$ and $a_2 \in W_2$. Then $a_1 \in W_1 \subset W$, $a_2 \in W_2 \subset W$. So we have $a_1 + a_2 \in W$. Hence $W_1 + W_2 \subset W$.

24.Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

Proof. First, every element $(a_1, a_2, \cdots, a_n) \in F$ can be written as $(a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_{n-1}, 0) + (0, 0, \cdots, a_n) \in W_1 + W_2$. Hence $F^n = W_1 + W_2$. On the other hand, if $(a_1, a_2, \cdots, a_n) \in W_1 \cap W_2$, then $a_1 = a_2 = \cdots = a_n = 0$. So $W_1 \cap W_2 = \{0\}$. We have $F^n = W_1 \oplus W_2$.

27.Let V denote the vector space consisting of all upper triangular $n \times n$ matrices, and let W_1 denote the subspace of V consisting of all diagonal matrices. Show that $V = W_1 \oplus W_2$, where $W_2 = \{A \in V : A_{ij} = 0 \text{ whenever } i \geq j\}$.

Proof. As in 24. Every upper triangular matrix can be written as a sum of elements in W_1 and W_2, so $V = W_1 + W_2$. If there is a matrix A in $W_1 \cap W_2$, then every entry of A must be 0, that means $W_1 \cap W_2 = \{0\}$. So $V = W_1 \oplus W_2$.