MATH2007 – Notes — By Eric Hua

Contents

4.4 Indeterminate Forms and L'Hospital's Rule

In this section, we are going to deal with the limit with the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad 0\cdot\infty, \quad 0^0, \ldots$$

L'Hospital's rule: If $\frac{f(x)}{g(x)}$ becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to x_0$, where x_0 is finite or ∞, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Remark. $x \to x_0$ can be replaced by any of the symbols $x \to x_0^+$, $x \to x_0^-$, $x \to \infty$, or $x \to -\infty$.

Example 1 *Calculate*

$$\lim_{x\to 0} \frac{\sin x}{x}, \quad \lim_{x\to 0} \frac{\sin x}{x^2}, \quad \lim_{t\to 0} \frac{e^t - t - 1}{t^2}, \quad \lim_{x\to\infty} \frac{(\ln x)^2}{x^2}.$$

Example 2 *Calculate*

$$\lim_{x\to\infty} x^2 e^{-x}, \quad \lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x.$$

Solution:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \lim_{x\to\infty} \frac{0}{e^x} = 0.$$

To solve the second limit, let $y = \left(1 - \frac{1}{x}\right)^x$, then

$$\ln y = x \ln\left(1 - \frac{1}{x}\right) = \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}}.$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{-\frac{1}{x^2}\left(1 - \frac{1}{x}\right)} = -1. \Rightarrow$$

$$\lim_{x\to\infty} y = \frac{1}{e}.$$

5.5 The Substitution Rule

Two substitution rules:

- $$\int f(g(x))g'(x)dx = \int f(u)du, \quad u = g(x).$$

  In the final result, we have to replace u by $g(x)$;

- $$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 3 *Evaluate*

$$\int (2x-1)(x^2-x)^{100}\,dx.$$

Solution: Let $u = x^2 - x$. Then $du = (2x-1)dx$. Thus

$$\int (2x-1)(x^2-x)^{100}\,dx = \int u^{100}du = \frac{u^{101}}{101} + C = \frac{(x^2-x)^{101}}{101} + C.$$

Example 4

$$\int x\sqrt{x^2+1}\,dx.$$

Solution: Let $u = x^2 + 1$. Then $du = (2x)dx$. Thus

$$\int x\sqrt{x^2+1}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

Example 5

$$\int_0^1 x\sqrt{x^2+1}\,dx.$$

Solution: Let $u = x^2 + 1$. Then $du = (2x)dx$, $x = 0 \leftrightarrow u = 1$, $x = 1 \leftrightarrow u = 2$. Thus

$$\int_0^1 x\sqrt{x^2+1}\,dx = \int_1^2 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^2 = \frac{2\sqrt{2}-1}{3}.$$

Example 6 *Find*

$$\int \frac{1}{e^{-x}+1}dx.$$

Solution:

$$\int \frac{1}{e^{-x}+1}dx = \int \frac{e^x}{1+e^x}dx.$$

Let $u = 1 + e^x$, then $du = e^x dx$.

$$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|1+e^x| + C = \ln(1+e^x) + C.$$

Example 7 *Evaluate*

$$\int x^2 e^{x^3+1}\, dx.$$

Solution: Let $u = x^3 + 1$.

Example 8 *Evaluate*

$$\int \frac{x}{\sqrt{1-x}}\, dx.$$

Example 9 *Calculate*

$$\int \tan x\, dx.$$

Solution: Let $u = \sin x$.

Even and odd functions: If $f(-x) = -f(x)$ for all x, then $f(x)$ is odd; if $f(-x) = f(x)$ for all x, then $f(x)$ is even.

- If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$;
- If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Example 10 $\int_{-2}^{2} xe^{5x^4+x^2}dx = 0.$

7.1 Integration by Parts

Integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Example 11 *Evaluate*

$$\int (x^2 + x + 1)e^x\, dx.$$

Solution: Integration by parts

$$\int (x^2+x+1)e^x\,dx = \int (x^2+x+1)\,de^x = (x^2+x+1)e^x - \int (2x+1)e^x\,dx$$

$$= (x^2+x+1)e^x - \int (2x+1)\,de^x = (x^2+x+1)e^x - (2x+1)e^x + 2\int e^x\,dx$$

$$= (x^2+x+1)e^x - 2xe^x + 2e^x + C$$

Example 12 *Evaluate*

$$\int 4x^3 \ln x\,dx.$$

Solution: Integration by parts

$$\int 4x^3 \ln x\,dx = \int \ln x\,dx^4 = x^4 \ln x - \int x^4\,d(\ln x)$$

$$x^4 \ln x - \int x^4\,\frac{dx}{x} = x^4 \ln x - \int x^3\,dx$$

$$= x^4 \ln x - \frac{1}{4}x^4 + C$$

Example 13 *Evaluate*

$$\int_0^1 \arctan x\,dx.$$

Sol: Step 1: Calculate

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = x\arctan x - \frac{1}{2}\ln(1+x^2).$$

Step 2:

$$\int_0^1 \arctan x\,dx = [x\arctan x - \frac{1}{2}\ln(1+x^2)]|_0^1 = \arctan 1 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Example 14 *Evaluate*

$$\int e^x \cos x\,dx.$$

7.2 Trigonometric Integrals

Method to solve

$$\int \sin^m x \cos^n x dx.$$

- If m is odd, then let $u = \cos x$.
- If n is odd, then let $u = \sin x$.
- If m and n are even, then use half-angle formula.

Example 15 *Evaluate*

$$\int \sin^2(x)\cos^3 x\, dx.$$

Answer: Use $\cos^2(x) = 1 - \sin^2(x)$ and then the substitution $u = \sin(x)$, $du = \cos(x)\, dx$:

$$\begin{aligned}
\int \sin^2(x)\cos^3 x\, dx &= \int \sin^2(x)(1-\sin^2(x))\cos(x)\, dx \\
&= \int u^2(1-u^2)\, du \\
&= \int u^2 - u^4\, du \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c \\
&= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + c
\end{aligned}$$

Example 16 *Evaluate*

$$\int \sin^2(x)\, dx.$$

Sol: Use $\sin^2 x = \frac{1-\cos 2x}{2}$.

Example 17 *Evaluate*

$$\int \sin^2(x)\cos^4 x\, dx.$$

7.3 Trigonometric Substitution

Trigonometric substitutions:

1. $\sqrt{a^2 - x^2}$: Let $x = a\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$;

2. $\sqrt{a^2 + x^2}$: Let $x = a\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$;

3. $\sqrt{x^2 - a^2}$: Let $x = a\sec t$, $0 \le t < \frac{\pi}{2}$ or $\pi \le t < \frac{3\pi}{2}$.

Example 18 *Evaluate*

$$\int \sqrt{4 - x^2}\, dx.$$

Solution: Let $x = 2\sin\theta$. Then

$$\sqrt{4 - x^2} = 2\cos\theta, \quad dx = 2\cos\theta\, d\theta.$$

Hence

$$\begin{aligned}\int \sqrt{4 - x^2}\, dx &= \int (2\cos\theta)(2\cos\theta\, d\theta) \\ &= 4\int \cos^2\theta\, d\theta.\end{aligned}$$

Note that

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2},$$

hence

$$\begin{aligned}\int \sqrt{4 - x^2}\, dx = 4\int \cos^2\theta\, d\theta = 4\int \frac{1 + \cos 2\theta}{2}\, d\theta &= 2\int (1 + \cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta + C \\ &= 2arc\sin\frac{x}{2} + \sin\left(2arc\sin\frac{x}{2}\right) + C.\end{aligned}$$

Example 19 *Evaluate*

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1 - x^2}}\, dx.$$

Solution: Let $x = \sin t$. Then $x = 0 \leftrightarrow t = 0$, $x = 1/\sqrt{2} \leftrightarrow t = \pi/4$ and

$$\frac{x^2}{\sqrt{1 - x^2}} = \frac{\sin^2 t}{\cos t}, \quad dx = \cos t dt.$$

Therefore

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1 - x^2}}\, dx = \int_0^{\pi/4} \frac{\sin^2 t}{\cos t} \cos t dt$$

$$= \int_0^{\pi/4} \sin^2 t dt = \int_0^{\pi/4} \frac{1-\cos 2t}{2} dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/4} = \boxed{\tfrac{\pi}{8} - \tfrac{1}{4}}.$$

Example 20 *Evaluate*

$$\int \frac{1}{\sqrt{4+x^2}}\, dx$$

Answer: Let $x = 2\tan t$, then $dx = 2\sec t dt$.

7.4 Integration of Rational Functions by Partial Fractions

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}.$$

By long division,

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $P(x)$, $Q(x)$, $S(x)$ and $R(x)$ are polynomials, $\deg R < \deg Q$.
Partial fractions:

- If $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, a product of distinct linea factors, then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

- If some linear factors are same, for example, $Q(x) = (a_1x+b_1)^n(a_2x+b_2)\cdots(a_kx+b_k)$, then

$$\frac{R(x)}{Q(x)} = \frac{A_{11}}{a_1x+b_1} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1n}}{(a_1x+b_1)^n} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

- If $Q(x)$ has an irreducible factor ax^2+bx+c without repeating, then

$$\frac{R(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c} + \cdots.$$

- If $Q(x)$ has an irreducible factor ax^2+bx+c with repeating, e.g., $(ax^2+bx+c)^n$, then

$$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n} + \cdots .$$

Example 21 *Evaluate*

$$\int \frac{4x+5}{(2x+3)(3x+4)}\,dx$$

Solution: Let

$$\frac{4x+5}{(2x+3)(3x+4)} = \frac{A}{2x+3} + \frac{B}{3x+4} = \frac{(3A+2B)x+4A+3B}{(2x+3)(3x+4)}.$$

Hence

$$4x+5 = (3A+2B)x+4A+3B, \Rightarrow 4 = 3A+2B, 5 = 4A+3B, \Rightarrow A = 2, B = -1.$$

Thus

$$\int \frac{4x+5}{(2x+3)(3x+4)}\,dx = \int \left(\frac{2}{2x+3} + \frac{-1}{3x+4} \right) dx = \ln|2x+3| - \frac{1}{3}\ln|3x+4| + C.$$

Example 22 *Evaluate*

$$\int \frac{x^3+2x^2-5x}{x^2+2x-8}\,dx$$

Solution: Since

$$\frac{x^3+2x^2-5x}{x^2+2x-8} = x + \frac{3x}{x^2+2x-8} = x + \frac{3x}{(x+4)(x-2)} = x + \frac{2}{x+4} + \frac{1}{x-2}.$$

Hence

$$\begin{aligned}
\int \frac{x^3+2x^2-5x}{x^2+2x-8}\,dx &= \int \left(x + \frac{2}{x+4} + \frac{1}{x-2} \right) dx \\
&= \frac{1}{2}x^2 + \int \frac{2}{x+4}\,dx + \int \frac{1}{x-2}\,dx \\
&= \frac{1}{2}x^2 + 2\ln|x+4| + \ln|x-2| + C.
\end{aligned}$$

Example 23 *Evaluate*

$$\int \frac{4x^2+8x}{(x+1)^2(x-1)}\,dx$$

Solution: Let

$$\frac{4x^2+8x}{(x+1)^2(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x-1}, \Rightarrow A=1, B=2, C=3.$$

Hence

$$\int \frac{4x^2+8x}{(x+1)^2(x-1)}\,dx = \int \left(\frac{1}{(x+1)} + \frac{2}{(x+1)^2} + \frac{3}{x-1}\right) dx$$
$$= \ln|x+1| - \frac{2}{x+1} + 3\ln|x-1| + C.$$

Example 24 *Evaluate*

$$\int \frac{3x^2+5x+2}{x(x^2+2x+2)}\,dx$$

Solution: Let

$$\frac{3x^2+5x+2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}, \Rightarrow A=2, B=1, C=1.$$

Hence

$$\int \frac{3x^2+5x+2}{x(x^2+2x+2)}\,dx = \int \left(\frac{2}{x} + \frac{x+1}{x^2+2x+2}\right) dx$$
$$= 2\ln|x| + \frac{1}{2}\ln(x^2+2x+2) + C.$$

7.8 Improper Integrals

Type I: Infinite Intervals

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx, \quad \int_{-\infty}^b f(x)dx = \lim_{t\to\infty}\int_t^b f(x)dx,$$
$$\int_{-\infty}^\infty f(x)dx = \int_c^\infty f(x)dx + \int_{-\infty}^c f(x)dx.$$

Definition: Integral is convergent (or divergent) $\Leftrightarrow$ Integral is a finite number (or ∞).

Example 25

$$\int_1^\infty \frac{1}{x^2}dx = 1.$$

Example 26

$$\int_1^\infty \frac{1}{x}dx = \infty.$$

Example 27

$$\int_{-\infty}^1 \frac{1}{1+x^2}dx = \frac{3\pi}{4}.$$

Example 28

$$\int_{-\infty}^\infty e^{-|x|}dx = 2.$$

Example 29 *Determine if the integral*

$$I = \int_2^\infty xe^{-x}dx$$

is convergent or divergent and evaluate if it is convergent.

Solution. $I = \lim_{t\to\infty}\int_2^t xe^{-x}\,dx$ (integration by parts: let $u = x$ and $dv = e^{-x}dx$)
$= \lim_{t\to\infty}\left(x(-e^{-x})|_2^t - \int_2^t -e^{-x}\,dx\right)$
$= \lim_{t\to\infty}(-xe^{-x} - e^{-x})|_2^t$
$= \lim_{t\to\infty} -(x+1)e^{-x}|_2^t$
$= \lim_{t\to\infty}(-(t+1)e^{-t} + 3e^{-2})$
$= 3e^{-2}$ (where $\lim_{t\to\infty}(t+1)e^{-t} = 0$ by L'Hospital's Rule).

Type 2: Discontinuous Integrands

If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx;$$

If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)dx = \lim_{t\to a}\int_a^b f(x)dx;$$

If $f(x)$ is **discontinuous** at c: $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 30

$$\int_2^3 \frac{1}{\sqrt{3-x}}dx = \lim_{t\to 3}\int_2^t \frac{1}{\sqrt{3-x}}dx = \lim_{t\to 3}[-2\sqrt{3-x}]_2^t = 2.$$

Example 31 *Determine if the integral*

$$\int_0^2 \frac{1}{x-1}dx$$

is convergent or divergent and evaluate if it is convergent.

Example 32 $\int_0^e \ln x dx = 0.$

p-Integral

Example 33

$$\int_1^\infty \frac{1}{x^p}dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1; \\ \text{divergent}, & \text{if } p \le 1. \end{cases}$$

Example 34

$$\int_0^1 \frac{1}{x^p}dx = \begin{cases} \frac{1}{1-p}, & \text{if } p < 1; \\ \text{divergent}, & \text{if } p \ge 1. \end{cases}$$

Comparison Test for Improper Integral

If $f(x)$ and $g(x)$ are continuous and $f(x) \ge g(x) \ge 0$ on $x \ge a$. Then

(i) $\int_a^\infty f(x)dx$ is convergent $\Longrightarrow \int_a^\infty g(x)dx$ is convergent;

(ii) $\int_a^\infty g(x)dx$ is divergent $\Longrightarrow \int_a^\infty f(x)dx$ is divergent.

Example 35

$$\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx = \text{convergent}.$$

$\because$

$$\frac{1}{\sqrt{x^3+1}} \le \frac{1}{x^{3/2}}.$$

Example 36

$$\int_8^\infty \frac{1+\sqrt{x}}{x-6}dx = \text{divergent}.$$

$\because$

$$\frac{1+\sqrt{x}}{x-6} \ge \frac{1}{\sqrt{x}}.$$

Example 37 *Determine whether the integral*

$$\int_2^\infty \frac{\cos^4 x}{e^x + \sin^2 x + 1} dx$$

is convergent or divergent.

Solution: Since

$$\frac{\cos^4 x}{e^x + \sin^2 x + 1} \leq \frac{1}{e^x} = e^{-x},$$

$$\int_2^\infty \frac{\cos^4 x}{e^x + \sin^2 x + 1} dx \leq \int_2^\infty e^{-x} dx = e^{-2}.$$

By Comparison Theorem, the original integral is convergent.

Example 38 *Determine whether the integral*

$$\int_0^1 \frac{1}{x^{1.9} \sin^2 x} dx$$

is convergent or divergent.

Solution: Let $y = 1/x$. Then

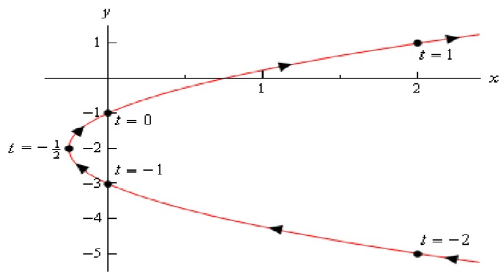
$$\int_0^1 \frac{1}{x^{1.9} \sin^2 x} dx = \int_1^\infty \frac{1}{y^{0.1} \sin^2 \frac{1}{y}} dy \geq \int_1^\infty \frac{1}{y^{0.1}} dy = \infty.$$

10.1 Curves defined by Parametric Equations

Instead of defining y in terms of x, $y = f(x)$, we define both x and y in terms of a third variable called a parameter as follows: $x = f(t), y = g(t)$. This third variable t is called a parameter. The collection of points $(x, y) = (f(t), g(t))$ that we get by letting t be all possible values is the graph of the parametric equations and is called the parametric curve.

Plane curve (parametric curve): The set of ordered pairs (points) $(x, y) = (f(t), g(t))$, where f and g are continuous functions on a parameter interval I.

Example 39 *Sketch the parametric curve for the following set of parametric equations.* $x = t^2 + t, y = 2t - 1$.



Remark. If you eliminate the parameter t, then $x = \frac{1}{4}(y+1)^2 + \frac{1}{2}(y+1)$.

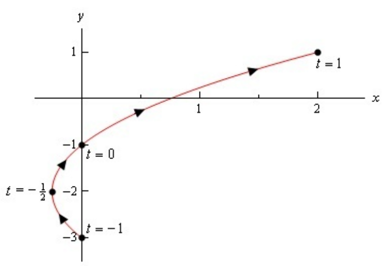
Now consider the following parametric curve with limits on the parameter:

$$x = f(t), y = g(t), a \leq t \leq b.$$

Then $(f(a), g(a))$ is called initial point, $(f(b), g(b))$ is called terminal point.

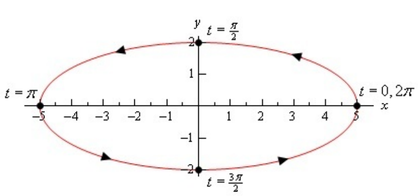
Example 40 *Sketch the parametric curve for the following set of parametric equations:*

$$x = t^2 + t, y = 2t - 1, -1 \leq t \leq 1.$$



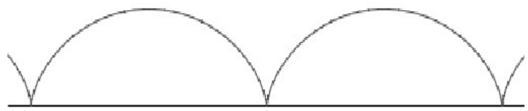
Example 41 *Sketch the parametric curve for the following set of parametric equations:*

$$x = 5\cos t, y = 2\sin t, 0 \leq t \leq 2\pi.$$



Example 42 *Sketch the Cycloid :*

$$x = r(t - \sin t), y = r(1 - \cos t), , -\infty < t < \infty.$$



10.2 Calculus with Parametric Curves

Tangents

We want to find the tangent lines to the parametric equations given by, $x = f(t), y = g(t)$. By Chain Rule, we have

- First Derivative for Parametric Equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \ provided \ \frac{dx}{dt} \neq 0.$$

- Second Derivative for Parametric Equations:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \ provided \ \frac{dx}{dt} \neq 0.$$

Example 43 *Find the tangent line(s) to the parametric curve given by*

$$x = t^5 - 4t^3, y = t^2, \ at \ (0, 4).$$

Solution. At first we need the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}.$$

When $(x, y) = (0, 4)$, $t = \pm 2$. At $t = -2$, the slope of the tangent line is: $-1/8$. The tangent line at $t = -2$ is: $y = 4 - x/8$; At $t = 2$, the slope is: $1/8$. The tangent line (at $t = 2$) is: $y = 4 + x/8$.

Example 44 *Find the points where the following parametric equations will have horizontal or vertical tangents:*

$$x = t^3 - 3t, y = 3t^2 - 9.$$

Solution. Horizontal Tangents: $dy/dt = 0, 6t = 0, t = 0$. Therefore, the only horizontal tangent will occur at the point $(x, y) = (0, -9)$.

Vertical Tangents: $dx/dt = 0$ (dy/dx undefined). In this case we need to solve, $3(t^2-1) = 0, \Rightarrow t = 1, -1$. The two vertical tangents will occur at the points (2,-6) and (-2,-6).

Example 45 *Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.*

$$x = 1 - t^2, y = t^7 + t^5.$$

Solution: To study concavity, we need the second derivative.

$$\frac{dy}{dx} = -\frac{7t^5 + 5t^3}{2} \Rightarrow \frac{d^2y}{dx^2} = \frac{35t^3 + 15t}{4}.$$

From $\frac{d^2y}{dx^2} = 0$ we imply that t=0. When $t < 0$, $\frac{d^2y}{dx^2} < 0$, the parametric curve will be concave down; when $t > 0$, $\frac{d^2y}{dx^2} > 0$, the parametric curve will be concave up.

Area*

Here we will find a formula for determining the area under a parametric curve given by the parametric equations:

$$x = f(t), y = g(t), \alpha \leq t \leq \beta.$$

We assume that the curve is traced out exactly once as t increases from α to β. As we know, the area under the curve $y = F(x), a \leq x \leq b$ is

$$A = \int_a^b y(x)\,dx = \int_\alpha^\beta y(t)x'(t)dt, \;\; where \;\; x(\alpha) = a, x(\beta) = b.$$

Example 46 *Find the area under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t).$$

Solution: $y = 0 \Rightarrow t = n\pi$. Thus

$$A = \int_\alpha^\beta y(t)x'(t)dt = \int_0^{2\pi} r^2(1-\cos t)^2 dt = r^2 \int_0^{2\pi} (1 - 2\cos t + \frac{1+\cos 2t}{2})dt = 3\pi r^2.$$

Arc Length

Here we will find a formula for determining the arc length to a parametric curve given by the parametric equations:

$$x = f(t), y = g(t), \alpha \leq t \leq \beta.$$

We assume that the curve is traced out exactly once as t increases from α to β. Also, for the purposes of the derivation that we're going to use, we will assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$dx/dt \geq 0, \alpha \leq t \leq \beta.$$

The arc length formula is given by

$$L = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2} dt, \;\; where \;\; x(\alpha) = a, x(\beta) = b.$$

Example 47 *Find the length under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t), \;\; 0 \leq t \leq 2\pi.$$

Solution:

$$L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_0^{2\pi} \sqrt{[r(1 - \cos t)]^2 + [r \sin t]^2} dt$$

$$= r \int_0^{2\pi} \sqrt{2 - 2\cos t} dt = r \int_0^{2\pi} 2 \left| \sin \frac{t}{2} \right| dt = 8r.$$

Example 48 *Find the length under one arch of cycloid:*

$$x = e^t + e^{-t}, y = 5 - 2t, \;\; 0 \leq t \leq 3.$$

Surface Area

Here we will determine the surface area of a region obtained by rotating a parametric curve about the x-axis or y-axis. We will rotate the parametric curve given by

$$x = f(t), y = g(t), \alpha \leq t \leq \beta$$

about the x-axis or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β.

(1) Area of a surface obtained by rotating a curve about x-axis:

$$S = \int_a^b 2\pi y\, ds = \int_\alpha^\beta 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \;\; ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

(2) Area of a surface obtained by rotating a curve about y-axis: just need to change y to x:

$$S = \int_a^b 2\pi x\, ds = \int_\alpha^\beta 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \;\; ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

Example 49 *Find the surface area of one arch of cycloid rotating about x-axis:*

$$x = r(t - \sin t), y = r(1 - \cos t), \;\; 0 \le t \le 2\pi.$$

Solution:

$$S = \int_\alpha^\beta 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^{2\pi} 2\pi r(1 - \cos t)\sqrt{[r(1 - \cos t)]^2 + [r \sin t]^2}dt$$

$$= 2\pi r^2 \int_0^{2\pi} (1 - \cos t)\sqrt{2 - 2\cos t}dt = 8\pi r^2 \int_0^{2\pi} \sin^3 \frac{t}{2}dt = 16\pi r^2 \int_0^\pi \sin^3 \phi d\phi$$

$$= -16\pi r^2 \int_0^\pi (1 - \cos^2 \phi)d\cos\phi = \frac{64}{3}\pi r^2.$$

Example 50 *Show that the surface area of a sphere of radius r is $4\pi r2$.*

Solution: The sphere is obtained by rotating the semicircle

$$x = r\cos t, y = r\sin t, 0 \le t \le \pi$$

about the x-axis.

Example 51 *Find the surface area obtained by rotating the curve*

$$x = t^3, y = t^2, \;\; 0 \le t \le 1.$$

about x-axis.

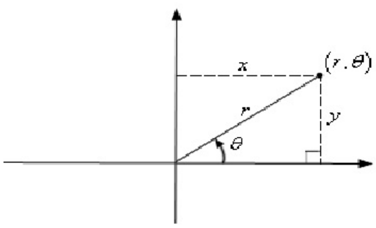
Solution:

$$S = \int_\alpha^\beta 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^1 2\pi t^3\sqrt{9t^2 + 4}dt$$

$$= 2\pi \int_4^{13} \frac{u - 4}{9}\sqrt{u}\frac{du}{18}.$$

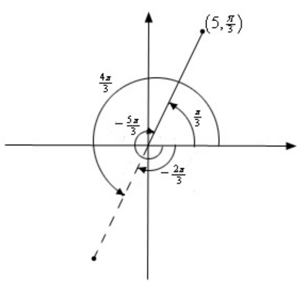
10.3 Polar Coordinates

We chose a point in the plane that is called the pole (or origin) and is labelled O. Then we draw a ray starting at O, along positive x-axis, which is called the polar axis. Let P be a point in the plane. Let r be the distance from P to O, let θ be the angle between OP and the polar axis. Then P can be represented by the ordered pair (r, θ). We call r and θ polar coordinates:



Agreement: $(-r, \theta) = (r, \theta + \pi)$.

Example 52 *Sketch of several polar coordinates:*



Polar ⇄ Cartesian Conversion Formulas:

$$x = r\cos\theta, y = r\sin\theta; \quad r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}.$$

Example 53 *Convert each of the following points into the given coordinate system.*
(a) $(-4, \frac{2\pi}{3})$ *into Cartesian coordinates. (b) (-1,-1) into polar coordinates.*

Solution: (a) $(x, y) = (2, -2\sqrt{3})$.
(b) $r = \sqrt{x^2 + y^2} = \sqrt{2}, \tan\theta = \frac{y}{x} = 1$. Since the point is in the third quadrant, the actual angle is, $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Polar curves

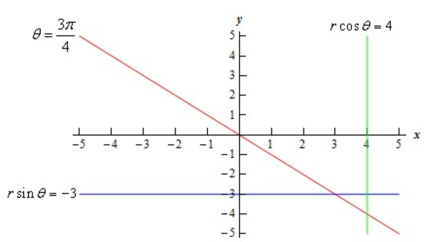
The graph of polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points $P(r, \theta)$ whose coordinates satisfy the equation. To sketch the graph, basically we need to combine

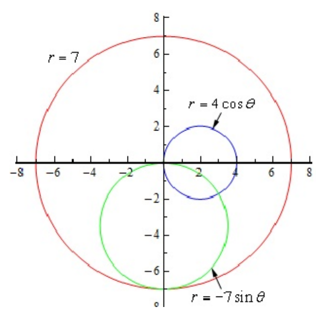
$$r = f(\theta), \quad x = r\cos\theta, y = r\sin\theta.$$

Some special cases:

- $\theta = \alpha$: This is a line that goes through the origin and makes an angle of α with the positive x-axis.
- $r\cos\theta = a$: This is equivalent to $x = a$.
- $r\sin\theta = b$: This is equivalent to $y = b$.
- $r = a$: A circle of radius a centered at the origin.
- $r = 2a\cos\theta$: A circle of radius $|a|$ and center $(a, 0)$.
- $r = 2b\sin\theta$: A circle of radius $|b|$ and center $(0, b)$.
- $r = 2a\cos\theta + 2b\sin\theta$: A circle of radius $r = \sqrt{x^2 + y^2}$ and center (a, b).

Example 54



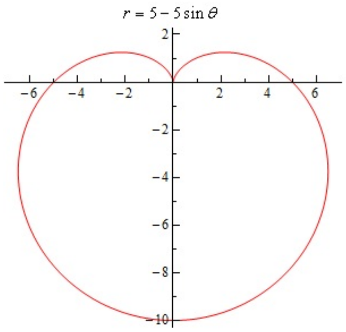


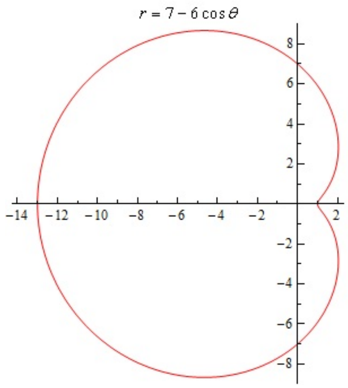
Remark. In the third graph we have an inner loop. To get this, we need to know the value of θ for which the graph will pass through the origin:

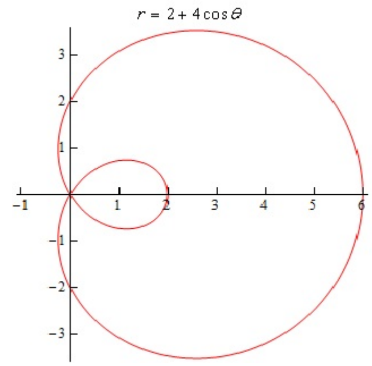
$$2 + 4\cos\theta = 0, \Rightarrow \cos\theta = -0.5, \Rightarrow \theta = 2\pi/3, 4\pi/3.$$

Proposition 1 *Suppose a polar curve is defined by $f(r, \theta) = 0$.*

1. *If $f(r, -\theta) = f(r, \theta)$, the graph is symmetric about the polar axis.*
2. *If $f(-r, \theta) = f(r, \theta)$, the graph is symmetric about the origin.*
3. *If $f(r, \pi - \theta) = f(r, \theta)$, the graph is symmetric about the line $\theta = \frac{\pi}{2}$.*







10.4 Areas and Lengths in Polar Coordinates

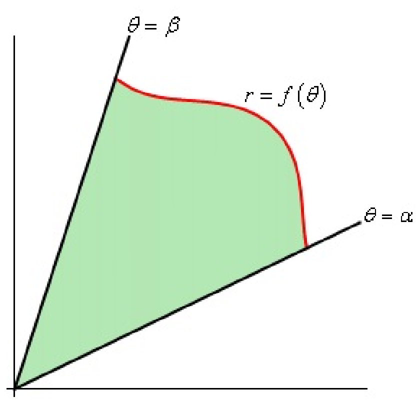
Area

As we know, the area of a sector with radius r and angle θ is

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

This implies that the area of the following polar region bounded by $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$ $(\alpha \le \beta)$ is:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta.$$



Example 55 *Find the area of the inner loop of* $r = 2 + 4\cos\theta$.

Solution: Let $r = 0$ we have:

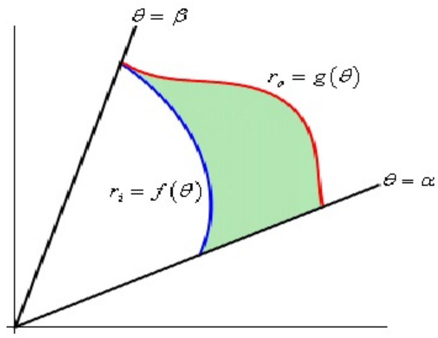
$$2 + 4\cos\theta = 0, \Rightarrow cos\theta = -0.5, \Rightarrow \theta = 2\pi/3, 4\pi/3.$$

So the inner loop is bounded by $r = 2 + 4\cos\theta$ and between $\theta = 2\pi/3$ and $4\pi/3$. Thus

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2 + 4\cos\theta)^2 d\theta.$$

$$= \int_{2\pi/3}^{4\pi/3} [6 + 8\cos\theta + 4\cos(2\theta)] d\theta = 4\pi - 6\sqrt{3}.$$

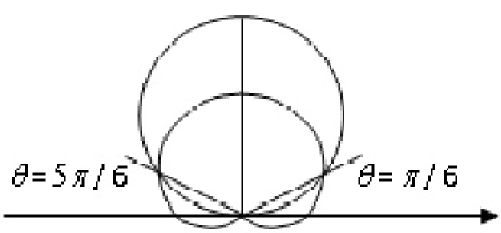
Now we consider a more general polar region:



The area of the shaded part will be :

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_i^2) d\theta.$$

Example 56 *Find the area of the part outside the cardioid* $r = 1 + \sin\theta$*, inside the circle* $r = 3\sin\theta$*.*



Solution: First find intersections: $1 + \sin\theta = 3\sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \pi/6, 5\pi/6$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_i^2) d\theta = \int_{\pi/6}^{5\pi/6} \frac{1}{2}[(3\sin\theta)^2 - (1 + \sin\theta)^2] d\theta = \pi.$$

Example 57 *Find the area of the part outside the cardioid* $r = 3 + 2\sin\theta$*, inside the circle* $r = 2$*.*

Solution: First find intersections: $3 + 2\sin\theta = 2 \Rightarrow \sin\theta = -1/2 \Rightarrow \theta = 7\pi/6, 11\pi/6$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_i^2) d\theta = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}[(2)^2 - (3 + 2\sin\theta)^2] d\theta = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}.$$

Arc length

Now we are going to find the formula for the arc length of the arc $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$ $(\alpha \leq \beta)$. Note that

$$x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta, \Rightarrow$$

$$(x'_\theta)^2 + (y'_\theta)^2 = (r'_\theta)^2 + r^2.$$

Thus

$$L = \int_\alpha^\beta \sqrt{(x'_\theta)^2 + (y'_\theta)^2} d\theta = \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2} d\theta.$$

Example 58 *Find the length of the spiral $r = \theta$, $0 \leq \theta \leq 1$.*

Solution:

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2} d\theta = \int_0^1 \sqrt{\theta^2 + 1} d\theta \\ &= \int_0^{\pi/4} \sec^3 x dx, \quad \theta = \tan x, d\theta = \sec^2 x dx \\ &= (\sec x \tan x + \ln|\sec x + \tan x|)|_0^{\pi/4} = \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}). \end{aligned}$$

Example 59 *Find the length of the cardioid $r = 1 + \sin\theta$, $0 \leq \theta \leq 2\pi$.*

Solution:

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2} d\theta = \int_0^{2\pi} \sqrt{2(\sin\theta + 1)} d\theta \\ &= \sqrt{2} \int_0^{2\pi} |\sin\frac{\theta}{2} + \cos\frac{\theta}{2}| d\theta = 2\sqrt{2} \int_0^\pi |\sin\varphi + \cos\varphi| d\varphi, \quad \varphi = \theta/2 \\ &= 2\sqrt{2} \int_0^{3\pi/4} (\sin\varphi + \cos\varphi) d\varphi - 2\sqrt{2} \int_{3\pi/4}^\pi (\sin\varphi + \cos\varphi) d\varphi \\ &= 8. \end{aligned}$$

11.1 Sequences

Sequence:

$$a_1, a_2, ..., a_n, ...$$

a_n is the nth term. If $\lim_{n\to\infty} a_n$ exists, then we say the sequence converges. Otherwise, we say the sequence diverges.

Example. The sequence $\left\{\frac{\ln n}{n}\right\}$ is convergent.

Solution:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Example. The sequence $\left\{\sqrt{n^2+3n-1} - \sqrt{n^2-1}\right\}$ converges to $\frac{3}{2}$.

Example. The sequence $\{\cos n\}$ is divergent; $\{\arctan(-n)\}$ converges to $-\frac{\pi}{2}$.

Example.

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r| < 1; \\ \infty, & |r| > 1. \end{cases}.$$

Thus the sequence $\left\{(\frac{1}{3})^n\right\}$ is convergent, but the sequence $\{(3)^n\}$ is divergent.

Property: If $a_n = f(n)$ and $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Properties: Let $\{a_n\}$ and $\{b_n\}$ be convergent, $c, d \in \mathbb{R}$.

1. $\lim\limits_{n\to\infty}(ca_n + db_n) = c\lim\limits_{n\to\infty} a_n + d\lim\limits_{n\to\infty} b_n$.
2. $\lim\limits_{n\to\infty}(a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$.
3. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $\lim\limits_{n\to\infty} b_n \neq 0$.
4. $\lim\limits_{n\to\infty} a_n^p = (\lim\limits_{n\to\infty} a_n)^p$, if $a_n \geq 0$ and $p > 0$.
5. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Squeeze Theorem: If $a_n \leq b_n \leq c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Example. $\left\{\frac{\cos n}{n}\right\}$ converges 0.

Bounded above: $a_n \le M$ for all n.
Bounded below: $a_n \ge m$ for all n.
Bound and Convergence: A convergent sequence is bounded.
Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.
Example. Given the sequence $\{a_n\}$: $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$.

(a) Show that the sequence is increasing and bounded above.

Solution: (i) We will show that $a_n < 6$ by induction: Note that $a_1 < 6$. Assume that $a_n < 6$. Then $a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6 + 6) = 6$.

(ii) $a_{n+1} - a_n = 3 - \frac{1}{2}a_n > 3 - \frac{6}{2} = 0$. Thus the sequence is increasing. (b) Find the limit of the sequence.

Solution: By Monotonic Sequence Theorem, the sequence is convergent. Let the limit be L. Then by the recursive relation, $L = \frac{1}{2}(L + 6) \Rightarrow L = 6$.

Example. $a_n = \frac{n}{n^3+1}$ is decreasing, and $0 < a_n < 1$ for any n.

To check it, let $f(x) = \frac{x}{x^3+1}$. Then $f'(x) = \frac{1-2x^2}{(x^3+1)^2} < 0$ when $x \ge 1$. Hence $f(x)$ is decreasing when $x \ge 1$.

11.2 Series

A basic fact is

$$\lim_{n\to\infty} r^n = 0 \Leftrightarrow |r| < 1.$$

The sum $\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + ...$ is called infinite series.
Partial sum:

$$S_n = \sum_{j=1}^{n} a_j.$$

Then

$$\sum_{j=1}^{\infty} a_j = S \Leftrightarrow \lim_{n\to\infty} S_n = S.$$

- Geometric series: if $|r| < 1$ then $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$.

- $\frac{k}{n(n+k)} = \frac{1}{n} - \frac{1}{n+k}$.
- Harmonic series $\sum_{j=1}^{\infty} \frac{1}{n}$ is divergent.
- Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{j=1}^{\infty} a_j$ is divergent.
- $\sum_{j=1}^{\infty}(ca_j + db_j) = c\sum_{j=1}^{\infty} a_j + d\sum_{j=1}^{\infty} b_j$.

Example. Determine if the series converges or diverges:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+2}}{3^{n+1}}$

(b) $\sum_{n=1}^{\infty} 3^{n+1}2^{-2n}$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{\ln n}$.

(d) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$.

Solution:(a)

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{2^{2(n+1)}}{3^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^{n+1}.$$

It is a geometric series with ratio $r = 4/3 > 1$. Therefore it is divergent.

(b) We have

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \sum_{n=1}^{\infty} 3 \cdot 3^n 4^{-n} = 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

This series is a geometric series with first term $a = 9/4$ and common ratio $r = 3/4$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \frac{a}{1-r} = \frac{9/4}{1/4} = 9.$$

(c) Divergent by Divergence Test.

(d)

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots\right) = \frac{5}{12}.$$

11.3 The Integral Test and Estimates of Sums

Integral test: If $f(x)$ is continuous, positive, decreasing, $f(j) = a_j$, then

$$\sum_{j=1}^{\infty} a_j \text{ is convergent} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ is convergent.}$$

Example. Determine if the series converges or diverges $\displaystyle\sum_{n=3}^{\infty} \frac{3}{n(\ln n)^3}$

Sol: We use Integral Test. Let $f(x) = \frac{3}{x(\ln x)^3}$. Then $f(x) > 0$ for $x \geq 3$. Since

$$f'(x) = \frac{-3[(\ln x)^3 + 3(\ln x)^2]}{x^2(\ln x)^6} < 0,$$

$f(x)$ is decreasing. By substitution with $u = \ln x$, we have

$$\int_3^{\infty} \frac{3}{x(\ln x)^3} dx = \lim_{b\to\infty} \int_3^b \frac{3}{x(\ln x)^3} dx$$

$$= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{3}{u^3} du$$

$$= \frac{3}{2}(\frac{1}{\ln 3})^2.$$

Hence it is convergent.

Example. Show that the series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges.

The $p-$series: $\sum_{j=1}^{\infty} \frac{1}{j^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example. For what value of p is the series

$$\sum_{n=1}^{\infty} n^{p-1}$$

convergent?

Solution:

$$\sum_{n=1}^{\infty} n^{p-1} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{-p+1}.$$

Hence, if $-p + 1 > 1$, i.e., $p < 0$, then the series is convergent.

Remainder estimate: Let $R_n = \sum_{j=n+1}^{\infty} a_j$. Then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx.$$

Example. (1) Estimate the error if we use the sum of the first 10 terms to approximate

$$\sum_{n=1}^{\infty} \frac{2}{n^3}.$$

What is the estimated sum? Give an interval to the sum.

(2) Find the smallest n such that S_n is within 0.000001 of the sum.

Solution. (1) Let $f(x) = \frac{2}{x^3}$. Then

$$\int_{11}^{\infty} f(x)dx \le R_{10} \le \int_{10}^{\infty} f(x)dx, \Rightarrow$$

$$\frac{1}{11^2} \le R_{10} \le \frac{1}{10^2}.$$

$$S_{10} = \sum_{n=1}^{10} \frac{2}{n^3} \doteq 2.395$$

$$2.395 - 0.01 \le S \le 2.395 + 0.01.$$

(2)

$$R_n \le \frac{1}{n^2}, \Rightarrow \frac{1}{n^2} \le 0.000001, \Rightarrow n \ge 1000,$$

$$R_{999} \ge \int_{1000}^{\infty} f(x)dx = 0.000001.$$

11.4 The Comparison Tests

Comparison test: if $0 \le a_j \le b_j$, then

$$\sum_{j=1}^{\infty} b_j \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

$$\sum_{j=1}^{\infty} a_j \text{ is divergent} \Rightarrow \sum_{j=1}^{\infty} b_j \text{ is divergent.}$$

Example. Determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{n(n^3+5)^{1/3}}$.

Solution: By Comparison test,

$0 \le \sum_{n=1}^{\infty} \frac{1}{n(n^3+5)^{1/3}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series, $p = 2 > 1$), so convergent.

The Limit Comparison test: Suppose that $a_j > 0$ and $b_j > 0$ for any j, and

$$\lim_{j \to \infty} \frac{a_j}{b_j} = c > 0.$$

Then either both $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are convergent or both diverge.

Example. Determine if the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{3n^3 + 2n + 1}{9n^3\sqrt{n} + 1}.$$

Solution. Note that

$$a_n = \frac{3n^3 + 2n + 1}{9n^3\sqrt{n} + 1} \sim \frac{3n^3}{9n^3\sqrt{n}} = \frac{1}{3\sqrt{n}}.$$

Let

$$b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}.$$

Then

$$\sum_{n=1}^{\infty} b_n$$

is divergent by p-series test with $p = 1/2$. Since

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{3},$$

the given series is divergent by Limit Comparison Test.

11.5 Alternating Series

1. Alternating series Test: The alternating series $\sum_{j=1}^{\infty} (-1)^{j-1} b_j$ is convergent if

1. $b_j > 0$
2. b_j decreasing $(b_1 \geq b_2 \geq b_3 \geq \cdots)$
3. $\lim_{j \to \infty} b_j = 0.$

For the alternating series, we can estimate the **remainder** by using the following inequality

$$|R_n| \le b_{n+1}.$$

Example. Test the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n^2+n+1}$ for convergence or divergence.

Solution: This is an alternating series. Let $b_n = \frac{n+1}{n^2+n+2}$. Then

1) $\lim_{n\to\infty} b_n = 0$;

2) Let $f(x) = \frac{x+1}{x^2+x+1}$ for $x \ge 1$, then

$$f'(x) = \frac{-x^2 - 2x}{(x^2+x+1)^2} < 0.$$

Therefore $f(x)$ decreases for $x \ge 1$. In particular,

$$f(n) > f(n+1)$$

for all positive integer n. Hence b_n decreases.

By the Alternating Series Test, the series converges.

11.6 Absolute Convergence and the Ratio and Root Tests

1. Absolute and Conditional Convergence If the series $\sum_{j=1}^{\infty}|a_j|$ is convergent, then we say that $\sum_{j=1}^{\infty} a_j$ is absolutely convergent; If the series $\sum_{j=1}^{\infty} a_j$ is convergent, but $\sum_{j=1}^{\infty}|a_j|$ is divergent, then we say that $\sum_{j=1}^{\infty} a_j$ is conditionally convergent.

Property:

$$\sum_{j=1}^{\infty}|a_j| \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

Example. Determine whether the series is absolutely convergent: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n^2+n+1}$.

Solution:

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n+2}{n^2+n+1}\right| = \sum_{n=1}^{\infty}\frac{n+2}{n^2+n+1}.$$

Note that
$$\frac{n+2}{n^2+n+1} \geq \frac{n}{3n^2} = \frac{1}{3n}.$$
The series
$$\sum_{n=1}^{\infty} \frac{1}{3n}$$
is divergent (the harmonic series), by Comparison Theorem, the original series is not absolutely convergent.

By Alternating Series Test, the series is convergent, therefore the series is conditionally convergent.

Example. Show that the series is absolutely convergent: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(5n)}{n^2+7n+3}$.

2. Ratio test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}| = L$.

- If $L < 1$, then the series is absolutely convergent;
- If $L > 1$, then the series is divergent.
- If $L = 1$, then the test is inconclusive.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ for convergence or divergence.

Solution: Let $a_n = \frac{n!}{n^n}$. Then
$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1.$$
By the Ratio Test, the series converges.

3. Root test: Consider the series $\sum_{n=1}^{\infty} a_n$, with $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

- If $L < 1$, then the series is absolutely convergent;
- If $L > 1$, then the series is divergent.
- If $L = 1$, then the test is inconclusive.

Example. Test the series $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ for convergence or divergence.

Solution: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$. By the Root Test, the series converges.

Example. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2\sqrt[n]{n}+1)^n}$ is convergence.

Solution: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2e^0+1} = \frac{1}{3} < 1$. By the Root Test, the series converges.

11.8 Power Series

Definition. The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called power series, a is called the center. There exists $R \geq 0$ such that the series is convergent in $|x-a| < R$ and divergent in $|x-a| > R$. R is called radius of convergence.

- Radius of convergence: Using Ratio Test to find R.
- Interval of convergence I: Symmetric to the center a, with two end points $a-R$ and $a+R$. The convergence or divergence at the two end points $x = a-R$ and $x = a+R$ should be checked.

Example. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n}$.

Sol: We use Ratio Test.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)\,5^{n+1}} / \frac{(-1)^n(x-1)^n}{n\,5^n}| = \lim_{n\to\infty} \frac{1}{5}\frac{n}{n+1}|x-1|$$

$$= \frac{1}{5}|x-1| = L.$$

When $L < 1$, we have $|x-1| < 5$. Hence

$$R = 5,$$

and $-4 < x < 6$.

When $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-4-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent (Harmonic series).

When $x = 6$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(6-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

which is convergent by Alternating Series Test. Therefore,

$$I = (-4, 6].$$

Example. Find the radius and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2}$$

Solution: Let

$$a_n = \frac{(2x-3)^n}{n^2}.$$

Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot |(2x-3)| = |2x-3|.$$

By Ratio Test, $|2x-3| < 1$, i.e., $|x-1.5| < 0.5$. Therefore $R = 0.5$ and $1 < x < 2$.

When $x = 1$,

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which is convergent by Alternating Series Test.

When $x = 2$,

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is convergent by $p-$Series Test. Therefore,

$$I = [1, 2].$$

Example. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{n\,8^n}$.

Sol: We use Ratio Test.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{1}{8} \frac{n}{n+1} |x-1|^3$$
$$= \frac{1}{8}|x-1|^3.$$

By Ratio Test, we have $|x-1|^3 < 8$, i.e., $|x-1| < 2$. Hence

$$R = 2, \quad and \quad -1 < x < 3.$$

When $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{n\,8^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent (Harmonic series).

When $x = 3$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{n\, 8^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

which is convergent by Alternating Series Test. Therefore,

$$I = (-1, 3].$$

11.9 Representations of Functions as Power Series

- Basic result:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

- Term-by-term differentiation: If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

and f and f' have the same radius of convergence.

- Term-by-term integration: If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then

$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1},$$

and f and $\int f dx$ have the same radius of convergence.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1.$

(ii) $\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}, \quad |-2x^2| < 1.$

(iii) $\frac{x^3}{6x+3} = \frac{x^3}{3}\frac{1}{1+2x} = \frac{x^3}{3}\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{3}(x)^{n+3}.$

To find the domain,

Method 1: This series is convergent when $|-2x| < 1$, which is $|x| < 0.5$. So $I = (-0.5, 0.5)$.

Method 2: Let $a_n = (-1)^n \frac{2^n}{3}(x)^{n+3}$. Then

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{2^{n+1}}{3}(x)^{n+4}}{(-1)^n\frac{2^n}{3}(x)^{n+3}}\right| = 2|x|.$$

By the Ratio Test, when $L < 1$, we have $2|x| < 1$, $|x| < 0.5$. Hence $R = 0.5$. When $x = \pm 0.5$, $(-1)^n\frac{2^n}{3}(x)^{n+3} \not\to 0$, the series diverge. So $I = (-0.5, 0.5)$.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $f(x) = \frac{1}{(1+x)^2}$.

$$\frac{1}{(1+x)^2} = \frac{d}{dx}(\frac{-1}{1+x}) = \frac{d}{dx}[(-1)\sum_{n=0}^{\infty}(-x)^n] = \sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}, \quad -1 < x < 1.$$

(ii) $f(x) = \frac{1}{(1-x)^3}$.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $f(x) = \ln(1+x)$.

Solution: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$, $R = 1$. $I = (-1, 1]$.

(ii) $f(x) = \arctan x$.

Solution:

$$\arctan x = \int \frac{1}{1+x^2}dx = \int (\sum_{n=0}^{\infty}(-1)^n x^{2n})dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad R = 1.$$

Note that $C = \arctan 0 = 0$, we have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad -1 \le x \le 1.$$

Remark. It's difficult to check the two end-points.

Example. (i) Represent $\int \frac{1}{1+x^3}dx$ as power series and use it to approximate $\int_0^{0.1} \frac{1}{1+x^3}dx$ correct to within 0.00001.

Example. Represent $\frac{3x-1}{x^2-1}$ as a power series.

Solution:

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1} = \sum_{n=0}^{\infty}[2(-1)^n - 1]x^n.$$

Example. Represent $\frac{1}{x}$ as a power series, centered at 1.

Solution:

$$\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n, \quad |x-1| < 1.$$

11.10 Taylor and MacLaurin Series

- Taylor series for $f(x)$ at the center a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

- Maclaurin series for $f(x)$ = Taylor series for $f(x)$ at the center 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n;$$

- Taylor polynomial:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

$$f(x) \approx T_n(x).$$

- Taylor's inequality: If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then on $|x-a| \le d$,

$$|R_n(x)| = |f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

- Taylor Theorem: If $f(x) = T_n(x) + R_n(x)$, and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then $f(x)$ is equal to its Taylor series for $|x-a| < R$. To this end, the following result is useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

- Series for composite functions;

- Multiplication and division of power series.

Maclaurin series for some special functions

Example. Maclaurin series for some special functions:

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...,\ R = \infty;$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...,\ R = \infty;$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ..., R = \infty;$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...,\ R = 1;$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad R = 1.$

Series for composite functions

Example. Maclaurin series:

(i) e^{x^2}, $\sin(x^2)$.

(ii)

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + ...$$

$$= 1 + (x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...) + (\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^2}{2!}) + (\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^3}{3!}) + ...$$

$$= 1 + x + \frac{x^2}{2!} + 0x^3 + ...$$

Binomial series

If k is a real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} x^n,$$

here

$$\begin{pmatrix} k \\ n \end{pmatrix} = \frac{k(k-1)\cdots(k-n+1)}{n!}, \quad \begin{pmatrix} k \\ 0 \end{pmatrix} = 1.$$

Application: Let $f(x) = (1+x)^k$, then

$$f^{(n)}(0) = \begin{pmatrix} k \\ n \end{pmatrix} n! = k(k-1)\cdots(k-n+1).$$

Example. Maclauring series for $f(x) = \sqrt[3]{1+x}$.

Solution:

$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{2}{3^2 2!}x^2 + \frac{2 \cdot 5}{3^3 3!}x^3 - \frac{2 \cdot 5 \cdot 8}{3^4 4!}x^4 + \cdots .$$

Example. Let $f(x) = \sqrt[5]{1+x^2}$. Evaluate $f^{(4)}(0)$.

Solution. Use the binomial series to find the Maclaurin series of $f(x)$.

$$\frac{f^{(n)}(0)}{n!} = \begin{pmatrix} k \\ n \end{pmatrix}$$

Hence

$$f^{(n)}(0) = \begin{pmatrix} k \\ n \end{pmatrix} n! = k(k-1)\cdots(k-n+1).$$

So $f^{(4)}(0) = -0.8064$.

Taylor series at other centers

Example. Find the Taylor series for $f(x) = \sin x$ at the center $x = \frac{\pi}{3}$.

Multiplication and division of Taylor series

Example. Evaluate

$$\lim_{x \to 0} \frac{e^x \sin x - x}{x^2}.$$

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(x - \frac{x^3}{3!} + ...\right)$$

$$= x + x^2 + \frac{1}{3}x^3 + ...$$

Therefore

$$\frac{e^x \sin x - x}{x^2} = 1 + \frac{1}{3}x + ...$$

Example. Let

$$f(x) = \int_0^x t^3 e^{3t} dt$$

(a) Find the Maclaurin series of the function f.

(b) Find the radius of the series in (a).

Solution: a)

$$t^3 e^{3t} = t^3 \sum_{n=0}^{\infty} \frac{(3t)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} t^{n+3}.$$

Hence

$$f(x) = \int_0^x \sum_{n=0}^{\infty} \frac{3^n}{n!} t^{n+3} dt$$

$$= \sum_{n=0}^{\infty} \int_0^x \frac{3^n}{n!} t^{n+3} dt = \sum_{n=0}^{\infty} \frac{3^n}{n!} \frac{t^{n+4}}{n+4} |_0^x$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!(n+4)} x^{n+4}$$

b)

$$|a_{n+1}/a_n| = \frac{3(n+4)}{(n+1)(n+5)}|x| \to 0$$

as $n \to \infty$. Therefore $R = \infty$.

Example. Find the first 5 non-zero terms in the Maclaurin series for $e^x \cos(3x)$.

Solution. Note that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Hence

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + ...$$

$$= 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - ...$$

We have

$$e^x \cos(3x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$$

$$-\frac{9x^2}{2} - \frac{9x^3}{2} - \frac{9x^4}{4} - ...$$

$$+\frac{27x^4}{8} + ...$$

$$= 1 + x + (\frac{1}{2} - \frac{9}{2})x^2 + (\frac{1}{6} - \frac{9}{2})x^3 + (\frac{1}{24} - \frac{9}{4} + \frac{27}{8})x^4 + ...$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \frac{7}{6}x^4 + ...$$

Example. Find the first 3 non-zero terms in the Maclaurin series for $\tan x$.

Solution: $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$

Applications of Taylor series.

Example.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

Example. Calculate the sum of the following series, given that it is a known series evaluated at a certain value of x:

$$1 - \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} - \frac{1}{64 \cdot 6!} + \frac{1}{256 \cdot 8!} - \cdots$$

Sol:

$$1 - \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} - \frac{1}{64 \cdot 6!} + \frac{1}{256 \cdot 8!} - \cdots$$

$$= 1 - \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 - \frac{1}{6!}\left(\frac{1}{2}\right)^6 + \frac{1}{8!}\left(\frac{1}{2}\right)^8 - \cdots$$

$$= \cos\frac{1}{2}.$$

Taylor polynomials and approximations

Taylor polynomial of degree n approximating $f(x)$ for x at a:

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $n = 1$, we have the linear approximation.

Example. Taylor polynomials of $f(x) = \sin x$ at $x = 0$:

$$P_1(x) = f(a) + f'(a)(x-a) = f(0) + f'(0)(x-0) = \sin 0 + (\cos 0)x = x,$$

$$P_3(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3 = \sin 0+(\cos 0)x-\frac{\sin 0}{2!}x^2-\frac{\cos 0}{3!}x^3$$

$$= x - \frac{x^3}{6}.$$

Example. Let $f(x)$ be a function such that

$$f(1) = 0, f'(1) = \frac{1}{5}, f''(1) = \frac{1}{10}, f'''(1) = \frac{1}{25}.$$

Estimate $f(1.15)$ using the Taylor expansion with order 3.

$$f(1.15) \approx T_3(1.15) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(1.15-1)^n = 0.0311475$$

.

Error in Taylor polynomial approximation: Let $P_n(x)$ be the Taylor approximation of $f(x)$ at $x = a$, then Taylor's inequality (The Lagrange Error Bound): If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then on $|x-a| \le d$,

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Example. Give a bound on the error E_3 to the function e^{2x} about 0 for $-1 \le x \le 1$.

Sol: Let $f(x) = e^{2x}$.

$$P_3(x) = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}.$$

$$f^{(3+1)}(x) = f^{(4)}(x) = 2^4 e^{2x}, \Rightarrow |f^{(3+1)}(x)| \le 16e^2, |x| \le 1.$$

$$|E_3(x)| = |f(x) - P_3(x)| \le \frac{|f^{(3+1)}(x)|}{(3+1)!}|x-0|^{3+1} \le \frac{16e^2}{4!}1^{3+1} = 2e^2/3.$$