SECTION 1.1

1. Second order; linear

2. Third order; nonlinear because of $(dy/dx)^4$

3. Fourth order; linear

5. Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1+(dy/dx)^2}$

6. Second order; nonlinear because of R^2

8. Second order; nonlinear because of $\dot{x}^2$

10. Writing the differential equation in the form $u(dv/du)+(1+u)v=ue^u$ we see that it is linear in v. However, writing it in the form $(v+uv-ue^u)(du/dv)+u=0$, we see that it is nonlinear in u.

11. From $y=e^{-x/2}$ we obtain $y'=-\frac{1}{2}e^{-x/2}$. Then $2y'+y=-e^{-x/2}+e^{-x/2}=0$.

13. From $y=e^{3x}\cos 2x$ we obtain $y'=3e^{3x}\cos 2x-2e^{3x}\sin 2x$ and $y''=5e^{3x}\cos 2x-12e^{3x}\sin 2x$, so that $y''-6y'+13y=0$.

14. From $y=-\cos x\ln(\sec x+\tan x)$ we obtain $y'=-1+\sin x\ln(\sec x+\tan x)$ and $y''=\tan x+\cos x\ln(\sec x+\tan x)$. Then $y''+y=\tan x$.

21. Differentiating $P=c_1e^t/(1+c_1e^t)$ we obtain

$$\frac{dP}{dt}=\frac{(1+c_1e^t)\,c_1e^t-c_1e^t\cdot c_1e^t}{(1+c_1e^t)^2}=\frac{c_1e^t}{1+c_1e^t}\,\frac{[(1+c_1e^t)-c_1e^t]}{1+c_1e^t}$$

$$=\frac{c_1e^t}{1+c_1e^t}\left[1-\frac{c_1e^t}{1+c_1e^t}\right]=P(1-P).$$

23. From $y=c_1e^{2x}+c_2xe^{2x}$ we obtain $\dfrac{dy}{dx}=(2c_1+c_2)e^{2x}+2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2}=(4c_1+4c_2)e^{2x}+4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=(4c_1+4c_2-8c_1-4c_2+4c_1)e^{2x}+(4c_2-8c_2+4c_2)xe^{2x}=0.$$

24. From $y=c_1x^{-1}+c_2x+c_3x\ln x+4x^2$ we obtain

$$\frac{dy}{dx}=-c_1x^{-2}+c_2+c_3+c_3\ln x+8x,$$

$$\frac{d^2y}{dx^2}=2c_1x^{-3}+c_3x^{-1}+8,$$

and

$$\frac{d^3y}{dx^3}=-6c_1x^{-4}-c_3x^{-2},$$

so that

$$\begin{aligned}x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y&=(-6c_1+4c_1+c_1+c_1)x^{-1}+(-c_3+2c_3-c_2-c_3+c_2)x\\&\quad+(-c_3+c_3)x\ln x+(16-8+4)x^2\\&=12x^2.\end{aligned}$$

SECTION 1.2

In Problems 7–10, we use $x = c_1 \cos t + c_2 \sin t$ and $x' = -c_1 \sin t + c_2 \cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

7. From the initial conditions we obtain the system

$$c_1 = -1$$
$$c_2 = 8.$$

The solution of the initial-value problem is $x = -\cos t + 8 \sin t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2}$$
$$-\frac{1}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4) \cos t + (1/4) \sin t.$$

In Problems 11–14, we use $y = c_1 e^x + c_2 e^{-x}$ and $y' = c_1 e^x - c_2 e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^2 e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

17. For $f(x, y) = y^{2/3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}y^{-1/3}$. Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

SECTION 1.3

10. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3-2)$gal/min $= 1$ gal/min. After t minutes there are $300+t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300+t} \text{ lb/gal} \right) = \frac{2A}{300+t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}.$$

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}.$$

Using $A_h = \pi \left(\frac{2}{12}\right)^2 = \frac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100}\sqrt{64h} = -\frac{c\pi}{450}\sqrt{h}.$$