UNIVERSITY OF OTTAWA

Department of Economics **ECO 2145A** **Fall 2014**

Dr. M. Rafiquzzaman **Problem set # 3** **Solution**

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1. $P = 200 - Q_A - Q_B$; $TC_A = 1500 + 55Q_A + Q_A^2$; $TC_B = 1200 + 20Q_B + 2Q_B^2$

(a)

$\pi_A = PQ_A - TC_A = (200 - Q_A - Q_B)Q_A - 500 - 55Q_A - Q_A^2 = 200Q_A - Q_A^2 - Q_AQ_B - 1500 - 55Q_A^2 - Q_A^2$

$= 145Q_A - 2Q_A^2 - Q_AQ_B - 1500$

$\frac{\delta\pi_A}{\delta Q_A} = 145 - 4Q_A - Q_B = 0 \Rightarrow Q_A = \frac{1}{4}(145 - Q_B)$,.........(1.1), which is the best response function of A.

$\pi_B = PQ_B - TC_B = (200 - Q_A - Q_B)Q_B - 1200 - 20Q_B - 2Q_B^2 = 180Q_B - Q_AQ_B - 3Q_B^2 - 1200$

$\frac{\delta\pi_B}{\delta Q_B} = 180 - Q_A - 6Q_B = 0 \Rightarrow Q_B = \frac{1}{6}(180 - Q_A)$, ...(1.2), which is the best response function of B.

For Nash-Cournot equilibrium, solve (1.1) and (1.2). Substituting (1.2) into (1.1), we get

$$Q_A = \frac{1}{4}\left[145 - \frac{1}{6}(180 - Q_A)\right] = \frac{1}{24}[690 + Q_A] \Rightarrow 24Q_A - Q_A = \frac{690}{24} \Rightarrow Q_A = \frac{690}{23} = 30.$$

Substituting Q_A into (1.2), we get $Q_B = \frac{1}{6}(180 - 30) = 25$.

(b) Industry output $= Q = Q_A + Q_B = 30 + 25 = 55$;

Form the demand equation, Price = P $= 200 - 30 - 25 = \$145$

$$\pi_A = PQ_A - TC_A = (145)(30) - \left[1500 + 55(30) + (30)^2\right] = 4350 - 1500 - 1650 - 900 = \$300$$

$$\pi_B = PQ_B - TC_B = (145)(25) - \left[1200 + 20(25) + 2(25)^2\right] = 3625 - 1200 - 500 - 1250 = \$675$$

Industry profit $\pi = \pi_A + \pi_B = \$300 + \$675 = \$975$.

(c) (i)

$$\pi = \pi_A + \pi_B = 145Q_A - 2Q_A^2 - Q_AQ_B - 1500 + 180Q_B - Q_AQ_B - 3Q_B^2 - 1200$$

$$= 145Q_A - 2Q_A^2 - 2Q_AQ_B + 180Q_B - 3Q_B^2 - 300, \text{.........(1.3)}$$

$$\frac{\delta\pi}{\delta Q_A} = 145 - 4Q_A - 2Q_B = 0, \ldots\ldots\ldots\ldots(1.4)$$
$$\frac{\delta\pi}{\delta Q_B} = 180 - 2Q_A - 6Q_B = 0, \ldots\ldots\ldots\ldots(1.5)$$

For the cartel case, solve (1.4) and (1.5). From (1.4), $Q_A = \frac{1}{4}(145 - 2Q_B)$, ...(1.6)

Substituting (1.6) into (1.5), we get

$$180 - \frac{2}{4}[145 - 2Q_B] - 6Q_B = 0 \Rightarrow 720 - 290 + 4Q_B - 24Q_B = 0 \Rightarrow 20Q_B = 430 \Rightarrow Q_B = \frac{430}{20} = 21.5.$$

From (1.6), $Q_A = \frac{1}{4}[145 - (2)(21.5)] = \frac{102}{4} = 25.5.$

(c) (ii)

From the demand equation, $P = 200 - 21.5 - 25.5 = \$153$

$$\pi_A = PQ_A - TC_A = (153)(21.5) - \left[1500 + 55(25.5) + (25.5)^2\right] = 3901.5 - 1500 - 1402.5 - 650.25 = \$348.75$$
$$\pi_B = PQ_B - TC_B = (153)(21.5) - \left[1200 + 20(21.5) + 2(21.5)^2\right] = 3289.5 - 1200 - 430 - 924.50 = \$735.00.$$

Industry profit $= \pi = \pi_A + \pi_B = \$348.75 + \$735.00 = \$1083.75$

(c) (iii)

Additional profit made by Firm A $= \$348.75 - \$300 = \$48.75$
Additional profit made by Firm B $= \$735 - \$675 = 60$

Since the additional profits made by both firms are positive, the answer is: YES.

2. Market demand is: $p = 150 - y; y = y_1 + y_2$; $c(y) = 64 + 30y$;
$MC(y_1) = MC(y_2) = \$30$

(a).
From the demand function, the residual demand function of Firm 1 is: $p = (150 - y_2) - y_1$
$\Rightarrow MR_1 = (150 - y_2) - 2y_1$
Profit maximizing condition, $MR_1 = MC(y_1) \Rightarrow (150 - y_2) - 2y_1 = 30 \Rightarrow$
$y_1 = 60 - 0.5y_2$, ...(2.1),
which is the best response function of Firm 1.

Similarly, the residual demand function for Firm2 is: $p = (150 - y_1) - y_2$
$\Rightarrow MR_2 = (150 - y_1) - 2y_2$
Profit maximizing condition, $MR_2 = MC(y_2) \Rightarrow (150 - y_1) - 2y_2 = 30 \Rightarrow$
$y_2 = 60 - 0.5y_1$, ...(2.2)

which is the best response function of Firm 2.

For Nash-Cournot equilibrium, solve (2.1) and (2.2). Substituting (2.2) into (2.1) we have

$$y_1 = 60 - 0.5(60 - 0.5y_1) \Rightarrow 0.75y_1 = 30 \Rightarrow y_1 = \frac{30}{0.75} = 40.$$

From (2.2), $y_2 = 60 - 0.5(40) = 40.$

Industry output $y = y_1 + y_2 = 40 + 40 = 80$

From the market demand equation, $p = 150 - 80 = \$70$

$\pi_1 = py_1 - c(y_1) = 70(40) - 64 - 30(40) = 2800 - 64 - 1200 = \$1536$

By symmetry, $\pi_2 = \$1536$

(b) If Firm 1 and Firm 2 collude, they maximize joint profit.

$$\pi = \pi_1 + \pi_2 = py_1 - c(y_1) + py_2 - c(y_2) = (150 - y_1 - y_2)y_1 - 64 - 30y_1 + (150 - y_1 - y_2)y_2 - 64 - 30y_2$$

$$\Rightarrow \pi = 120y_1 - y_1^2 - 2y_1y_2 + 120y_2 - y_2^2 - 128, \quad \text{.................(2.3)}$$

$$\frac{\delta\pi}{\delta y_1} = 120 - 2y_1 - 2y_2 = 0 \Rightarrow 2(y_1 + y_2) = 120 \Rightarrow y_1 + y_2 = 60, \text{........} \qquad (2.4)$$

$$\frac{\delta\pi}{\delta y_2} = 120 - 2y_1 - 2y_2 = 0 \Rightarrow 2(y_1 + y_2) = 120 \Rightarrow y_1 + y_2 = 60, \text{.............} \quad (2.5)$$

For cartel solution, solve (2.4) and (2.5). Since both firms have identical costs, they share the market equally. From (2.4) and (2.5), $y_1 = y_2 = \frac{60}{2} = 30$.

Industry output $y = y_1 + y_2 = 30 + 30 = 60$.

From the demand equation $P = 150 - 60 = \$90$

Industry profit $\pi = py - c(y) = 90(60) - 64 - 30(60) = 5400 - 64 - 1800 = \$3536.$

They share the profit equally. $\pi_1 = \pi_2 = \frac{\$3536}{2} = \$1768$

(c)

In the Nash-Cournot situation, each firm earns \$1536.By contrast, in the collusion case, each firm earns \$1736. Thus each firm makes (\$1768 - \$1536) =\$232 less in the Nash-Cournot solution than in the collusion solution. This implies that the Nash-Cournot solution, although individually rational, is collectively irrational because each firm earns less in the N-C situation than it could earn in the collusion solution.

(d)

If there are n firms in the Nash-Cournot model, then Firm1's residual demand function is:

$p = (150 - y_2 - y_3 - ... - y_n) - y_1$ ....(2.6),

where $y_1, y_2, ..., y_n$ is the output of Firm1, Firm2, ..., and Firm n, respectively.

Profit maximizing condition of Firm1 is $MR_1 = MC_1$

From the residual demand equation, $MR_1 = (150 - y_2 - y_3 - ... - y_n) - 2y_1$; $MC_1 = 30$

$$MR_1 = MC_1 \Rightarrow (150 - y_2 - y_3 - ... - y_n) - 2y_1 = 30 \Rightarrow y_1 = 60 - \frac{(y_2 + y_3 + ... + y_n)}{2}, ..(2.7)$$

which is the best response function for Firm1. Since every firm has identical cost, in the Nash-Cournot equilibrium, the equilibrium level of output will be same for all firms, i.e.,

$y_1 = y_2 = y_3 = ... = y_n = y_0$ .....(2.8).

Substituting (2.8) into (2.7) we get

$$y_0 = 60 - \frac{(y_0 + y_0 + ... + y_0)}{2} = 60 - \frac{(n-1)y_0}{2} \Rightarrow 2y_0 = 120 - (n-1)y_0 \Rightarrow 2y_0 + (n-1)y_0 = 120 \Rightarrow$$

$$y_0(n+1) = 120 \Rightarrow y_0 = \frac{120}{(n+1)}$$

Thus in Nash-Cournot equilibrium, each firm produces $y_0 = \frac{120}{(n+1)}$..........(2.8)

So, industry output $y = ny_0 = \frac{120n}{(n+1)}$

Price $p = 150 - y = 150 - \frac{120n}{(n+1)} = \frac{150n + 150 - 120n}{n+1} = \frac{30n + 150}{n+1}$

Each firm's profit is: $\pi = py_0 - c(y_0) = \left(\frac{30n+150}{n+1}\right)\left(\frac{120}{n+1}\right) - 64 - 30y_0$

$$= py_0 - c(y_0) = \left(\frac{30n+150}{n+1}\right)\left(\frac{120}{n+1}\right) - 64 - 30\left(\frac{120}{n+1}\right) = \left(\frac{120}{n+1}\right)\left(\frac{30n+150}{n+1} - 30\right) - 64$$

$$= \left(\frac{120}{n+1}\right)\left(\frac{30n+150-30n-30}{n+1}\right) - 64 = \left(\frac{120}{n+1}\right)\left(\frac{120}{n+1}\right) - 64 = \left(\frac{120}{n+1}\right)^2 - 64$$

Thus, $\pi = \left(\frac{120}{n+1}\right)^2 - 64$.........(2.9)

In order to find out the number of firms in the free entry equilibrium, we first find the level of profit of each firm if there are $(n+1)$ firms in the Nash-Cournot model. When there are $(n+1)$ firms, each firm's profit is obtained by replacing "n" by $(n+1)$ in equation (2.9). Then each firm's profit is:

$$\hat{\pi} = \left(\frac{120}{n+1+1}\right)^2 - 64 = \left(\frac{120}{n+2}\right)^2 - 64 \text{....(2.10)}$$

In (2.10), n is the smallest integer such that $\hat{\pi} < 0$ (This implies that entry will not occur if profit is negative). From (2.10)

$$\hat{\pi} = \left(\frac{120}{n+2}\right)^2 - 64 < 0 \Rightarrow \left(\frac{120}{n+2}\right)^2 < 64 \Rightarrow \left(\frac{120}{n+2}\right) < 8 \Rightarrow (n+2) > \frac{120}{8} \Rightarrow (n+2) > 15$$

$\Rightarrow n > 13$. So, $n = 14$

3.17 (p.525, Textbook)

You can solve this problem using calculus or the formulas for the linear demand and constant marginal cost Cournot model from the chapter.

a. For the duopoly,
$q_1 = \frac{15-1+1}{3(1)} = 5$, $q_2 = \frac{15-1-2(1)}{3(1)} = 4$
$p^* = 15 - (q_1 + q_2) = 6$
$\pi_1 = (6-1)5 = 25$, $\pi_2 = (6-2)4 = 16$. Total output is $Q = 5 + 4 = 9$. Total profit is $\pi_d = 25 + 16 = 41$. Consumer surplus is $CS_d = 1/2(15-6)9 = 81/2 = 40.5$. At the efficient price (equal to marginal cost of 1), the output is 14. The deadweight loss is $DWL_d = 1/2(6-1)(14-9) = 25/2 = 12.5$.

b. A monopoly equates its marginal revenue and marginal cost: $MR = 15 - 2Q_m = 1 = MC$. Thus $Q_m = 7$, $p_m = 8$, $\pi_m = (8-1)7 = 49$. Consumer surplus is $CS_m = 1/2(15-8)7 = 49/2 = 24.5$. The deadweight loss is $DWL_m = 1/2(8-1)(14-7) = 49/2 = 24.5$.

c. The average cost of production for the duopoly is $[(5 \times 1) + (4 \times 2)]/(5 + 4) = 1.44$, whereas the average cost of production for the monopoly is 1. The increase in market power effect swamps the efficiency gain, so consumer surplus falls while deadweight loss nearly doubles.

4.1 (p. 526, Textbook)

a. Using Equation 14.16, the Cournot equilibrium quantity for each of the duopoly firms is $q = (a - m)/(3b) = (150 - 60)/3 = 30$. As a result, the Cournot price is $p = (a + 2m)/3 = (150 + 120)/3 = 90$ (using Equation 14.17).

b. From Equation 14.31, we know that the Stackelberg leader's quantity is $q_1 = (a - m)/(2b) =$
$(150 - 60)/2 = 45$. The follower's quantity, by substituting the expression for q_1 in Equation 14.31 into Equation 14.29, is $q_2 = (a - m)/(4b) = (150 - 60)/4 = 22.5$. Thus, the Stackelberg equilibrium price is $p = 150 - 45 - 22.5 = 82.5$.

5.3 (p. 526, Textbook)

Firm 1 wants to maximize its profit:

$$\pi_1 = (p_1 - 10)q_1 = (p_1 - 10)(100 - 2p_1 + p_2).$$

Its first-order condition is $d\pi_1/dp_1 = 100 - 4p_1 + p_2 + 20 = 0$, so its best-response function is $p_1 = 30 + \frac{1}{4}p_2$. Similarly, Firm 2's best-response function is $p_2 = 30 + \frac{1}{4}p_1$. Solving, the Nash-Bertrand equilibrium prices are $p_1 = p_2 = 40$. Each firm produces 60 units.

5.7 (p. 526, Textbook)

a. Wawa's profit function is: $\pi_W = (p_W - 2)(680 - 500p_W + 400p_S)$.

Sunoco's profit function is: $\pi_S = (p_S - 2)(680 - 500p_S + 400p_W)$.

The F.O.C. for Wawa is:

$$\frac{\partial \pi_W}{\partial p_W} = 680 - 500p_W + 400p_S - 500(p_W - 2) = 400p_S - 1000p_W + 1680 = 0. \qquad (1)$$

Wawa's best response is $p_w = 1.68 + 0.4p_s$

The F.O.C. for Sunoco is:

$$\frac{\partial \pi_S}{\partial p_S} = 680 - 500p_S + 400p_W - 500(p_S - 2) = 400p_W - 1000p_S + 1680 = 0. \qquad (2)$$

Sunoco's best response is $p_s = 1.68 + 0.4p_s$

Solving equations (1) and (2) simultaneously, we can obtain the Nash equilibrium prices:

$$\begin{cases} p_W^* = 2.8 \\ p_S^* = 2.8. \end{cases}$$

b. With the salty snacks, Wawa's profit function is:

$$\pi_W = (p_W + 0.25 - 2)(680 - 500p_W + 400p_S).$$

Sunoco's profit function is still: $\pi_S = (p_S - 2)(680 - 500p_S + 400p_W)$.

The F.O.C. for Wawa is now:

$$\frac{\partial \pi_W}{\partial p_W} = 680 - 500p_W + 400p_S - 500(p_W - 1.75) = 400p_S - 1000p_W + 1555 = 0. \qquad (3)$$

Wawa's best response is now $p_w = 1.555 + 0.4p_s$

The best response for Sunoco is still equation (2).

Solving equations (2) and (3) simultaneously, we can obtain the new Nash equilibrium prices:

$$\begin{cases} p_W^* = 2.65 \\ p_S^* = 2.74. \end{cases}$$