MATH 1107 R - Winter 2014 Final Practice Problems

Answers and solution sketches are for reference only.

They are by no means models of proper solutions for obtaining full marks.

1. In this question, $z = 1 + i$, $w = 2 + i$. Give the answer to each part. Simplify your answer as much as possible. All complex numbers must be in **rectangular/standard form**.

(a) $\overline{w} = 2 - i$

(b) $z - w = -1$

(c) $w^{-1} = \frac{2}{5} - \frac{1}{5}i$

(d) $|w| = \sqrt{5}$

(e) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & i \end{bmatrix}$. Then $\det(A^{10}) = -32i$.

(f) Let $B = \begin{bmatrix} 2 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$. The rank of B is 2.

(g) Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and $C = A^{\mathsf{T}} + 2B^{-1}$.

Then $C_{1,2} = 6$.

(h) The eigenvalues of $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ are 2 and 3.

(i) Let W be a proper subspace of the vector space of 2×3 matrices with real entries.

The largest possible value for the dimension of W is 5.

(j) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$ such that $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = 1$ and $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = -1$. Then $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = -\frac{5}{6}$.

2. Find all real values a such that the system of linear equations

$$\begin{array}{ccccccc} x & - & y & + & z & = & 1 \\ x & - & 2y & + & 2z & = & -1 \\ & & y & - & z & = & a \end{array}$$

has no solutions.

Solution 1 sketch. Subtracting the second equation from the first equation gives $y - z = 2$. Since the third equation has the same left-hand side, there is no solution if $a \neq 2$. If $a = 2$, the third equation is superfluous and system reduces to

$$\begin{array}{ccccccc} x & - & y & + & z & = & 1 \\ x & - & 2y & + & 2z & = & -1 \end{array}$$

which has $x = 3$, $y = 2$, $z = 0$ as a solution. So the system has no solution if and only if $a \neq 2$.

Solution 2 sketch. The augmented matrix for the system is

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & a \end{bmatrix}.$$

Transforming it using elementary row operations into row-echelon form gives

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & a-2 \end{bmatrix}.$$

So the system has no solution if and only if $a - 2 \neq 0$, or equivalently, $a \neq 2$.

3. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ be matrices defined over the real numbers.

(a) Find a basis for the nullspace of A.

Solution sketch. The RREF of A is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for the nullspace of A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) Find a basis for the column space of A.

Solution sketch. A basis for the column space of A is given by the first column of A.

(c) Determine if the nullspace of A is the same as the column space of B. Justify your answer.

Solution sketch. One can check that if b is a column of B, then $Ab = 0$. Hence, every column of B is in the nullspace of A, implying that the column space of B is a subspace of the nullspace of A. Since B has rank 2, the column space of B has dimension 2. Since the nullity of A is 2 as well, the column space of B is equal to the nullspace of A.

4. Consider the matrix

$$A = \begin{bmatrix} k & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & k & 1 \\ 1 & -1 & 2 & -1 \end{bmatrix}.$$

Determine all values of k such that A is singular.

Solution sketch. The determinant of A is $-k^2-k+3$ which is zero if and only if $k = \frac{-1\pm\sqrt{13}}{2}$. Hence, the values of k for which A is singular are $\frac{-1+\sqrt{13}}{2}$ and $\frac{-1-\sqrt{13}}{2}$.

5. Consider the linear transformation T from $\mathbb{R}^4$ to the vector space of 2×2 real matrices given by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2a + c & a - d \\ -2a + 2d & b - d \end{bmatrix}$$

(a) Determine a basis for the range of T.

Solution sketch. Note that the range of T is spanned by $\left\{ \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \right\}$.

However, the matrix $\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$ can be written as a linear combination of the other three, which are linearly independent. Hence, a basis for the range of T is $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \right\}$.

(b) Is T surjective? injective? bijective? Justify your answer.

Solution sketch. T is not surjective since the range of T has dimension 3 whereas the dimension of the codomain is 4.

T is not injective since its kernel contains the non-zero vector $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

T is not bijective since it is neither surjective nor injective.

6. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Determine A^{-1}.

Answer. $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$.

(b) Diagonalize A by finding a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Answer. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.