Phys 301

2014

Homework 3

1) a) $$\bar{p} = \hat{z}\int_0^{\pi} z\,(2\pi R\sin\theta\, R\,d\theta\ \sigma_0(1+\cos\theta))$$

$$z = R\cos\theta$$

$$\bar{p} = \hat{z}\ 2\pi R^3\sigma_0\int_0^{\pi}(1+\cos\theta)\cos\theta\, d\theta = \frac{4}{3}\pi R^3\sigma_0\,\hat{z}$$

b) $$\bar{p} = \hat{z}\int_0^{\pi}(z+R\cos\theta)\,2\pi R\sin\theta\, R\,d\theta\ \sigma_0(1+\cos\theta)$$

$$= \left(\frac{4}{3}\pi R^3\sigma_0 + 4\pi R^2\sigma_0 z\right)\hat{z}$$

$$= \frac{4}{3}\pi R^3\sigma_0\,\hat{z} + Qz\,\hat{z}$$

c) $$\bar{p} = \frac{4}{3}\pi R^3\sigma_0\,\hat{z} + Q\,(x\hat{x} + y\hat{y} + z\hat{z})$$

2)

$$V = \frac{Q}{4\pi\varepsilon_0 r'} \quad ; \quad r' = (r^2 + s^2 - 2sr\cos\theta)^{1/2}$$

$$\frac{r}{r'} = \frac{1}{\left(1 + \frac{s^2}{r^2} - 2\frac{s}{r}\cos\theta\right)^{1/2}}$$

$$V = \frac{Q}{4\pi\varepsilon_0 r}\left(\frac{r}{r'}\right) = \frac{Q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{2}\left(\frac{s^2}{r^2} - \frac{2s\cos\theta}{r}\right) + \frac{3}{8}\left(\frac{s^2}{r^2} - \frac{2s\cos\theta}{r}\right)^2 + \cdots\right]$$

$$V = \frac{Q}{4\pi\varepsilon_0 r}\left[1 + \frac{s}{r}\cos\theta + \frac{s^2}{r^2}\left(-\frac{1}{2} + \frac{3\cos^2\theta}{2}\right) - \cdots\right]$$

$$V_1 = \frac{Q}{4\pi\varepsilon_0 r} \quad , \quad V_2 = \frac{Qs\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$V_3 = \frac{Qs^2(3\cos^2\theta - 1)}{8\pi\varepsilon_0 r^3}$$

3) $$\mathcal{E} = \int_R^\infty \int_0^\pi \frac{\varepsilon_0 E^2}{2} r^2 \sin\theta \, dr \, d\theta (2\pi)$$

$$= \pi \varepsilon_0 \int_R^\infty \int_0^\pi \frac{p^2}{(4\pi\varepsilon_0 r^3)^2} (4\cos^2\theta + \sin^2\theta) \cdot (r^2 \sin\theta \, dr \, d\theta)$$

$$= \frac{p^2}{16\pi\varepsilon_0} \int_R^\infty \frac{dr}{r^4} \int_0^\pi (3\cos^2\theta + 1) \sin\theta \, d\theta$$

$$= \frac{p^2}{16\pi\varepsilon_0} \frac{1}{3R^3} \left[-\frac{3\cos^3\theta}{3} - \cos\theta \right]$$

$$= \frac{p^2}{12\pi\varepsilon_0 R^3}$$

4)
$$\mathcal{E}' = \int_{R_1}^{R_2} \frac{\varepsilon_0 E^2}{2} 2\pi \rho \, d\rho$$

$$= \int_{R_1}^{R_2} \left(\frac{\lambda}{2\pi\varepsilon_0 \rho}\right)^2 \rho \, d\rho = \frac{\lambda^2}{4\pi\varepsilon_0} \ln \frac{R_2}{R_1}$$

$$V = \int_{R_1}^{R_2} \frac{\lambda}{2\pi\varepsilon_0 \rho} d\rho = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{R_2}{R_1}$$

$$\Rightarrow \quad \lambda = \frac{2\pi\varepsilon_0 V}{\ln(R_2/R_1)}$$

$$\mathcal{E}' = \frac{1}{4\pi\varepsilon_0} \left[\frac{2\pi\varepsilon_0 V}{\ln R_2/R_1}\right]^2 \ln \frac{R_2}{R_1} = \frac{\pi\varepsilon_0 V^2}{\ln R_2/R_1}$$

Units: $[\mathcal{E}'] = \left[\frac{\text{farad}}{m} V^2\right] = \left[\frac{1}{m} \cdot \frac{C}{V} V^2\right] =$

$$\left[\frac{C \cdot V}{m}\right] = \left[\frac{\text{joule}}{m}\right]$$

$$\frac{R_2}{R_1} = \frac{5}{3}, \quad V = 5 \quad \Rightarrow \quad \mathcal{E}' = 1.36 \times 10^{-9} \ \frac{\text{joule}}{m}$$

5) Initial energy

a) $$\mathcal{E}_0 = \int_a^b \frac{\varepsilon_0 E^2}{2} 4\pi r^2 dr = \frac{Q^2}{8\pi\varepsilon_0} \int_a^b \frac{dr}{r}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Final energy

$$\mathcal{E}_1 = \frac{\varepsilon_0}{2}\int_a^\infty \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$+ \frac{\varepsilon_0}{2}\int_b^\infty \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right)$$

b) The extra field energy comes from the mechanical work done in separating the charges

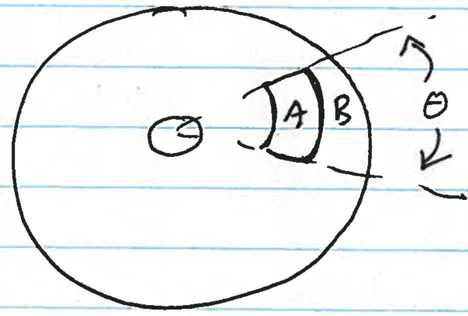
6) a) $V = \dfrac{Q}{4\pi \varepsilon R}$, $E = \dfrac{Q}{4\pi \varepsilon_0 R^2}$

$$V = ER = 3 \times 10^6 \times \frac{0.1}{2} = 150 \text{ kV}$$

b) $\dfrac{\varepsilon_0 E^2}{2} = 8.85 \times 10^{-12} \cdot \dfrac{9 \times 10^{12}}{2} = \dfrac{39.8 \text{ N}}{m^2}$

$$= \frac{39.8}{1.01 \times 10^5} \approx 4 \times 10^{-4} \text{ atm}$$

c) Enough to support a fabric with mass density $\sim 4 \dfrac{kg}{m^2}$

7) 

Consider a volume τ of dielectric with shape shown in the figure and length L.

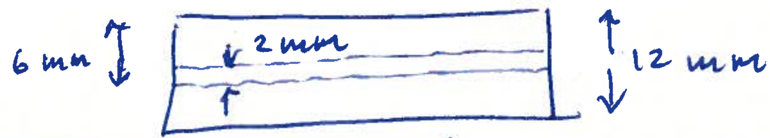
$$\int_\tau \nabla \cdot \bar{E}\, d\tau = \int_S \bar{E} \cdot d\bar{a} \qquad \text{where } S \text{ bounds } \tau.$$

$\bar{E} \cdot d\bar{a}$ is non-zero only on surfaces A and B.

If their radii are r_A and r_B

$$\int \bar{E} \cdot d\bar{a} = \frac{-\lambda \quad r_A \theta L}{2\pi \varepsilon_r \varepsilon_0 r_A} + \frac{\lambda \quad r_B \theta L}{2\pi \varepsilon_r \varepsilon_0 r_B} = 0$$

So, $\int_\tau \nabla \cdot \bar{E}\, d\tau = 0$, since r_A; r_B and θ are arbitrary, $\nabla \cdot \bar{E} = 0$.



8) $0.1\ \mu A$ on $25\ cm^2$ in $s = 2\ mm$

a) $$\rho_f = \frac{-0.6 \times 10^{12} \times 1.6 \times 10^{-19}}{2.5 \times 10^{-4} \times 2 \times 10^{-3}} = -0.02\ \frac{C}{m^3}$$

$$\rho_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\rho_f = 1.4 \times 10^{-2}\ C/m^3$$

b) $D = \frac{\sigma}{2}$ (Gauss's law) ; $P = \left(1 - \frac{1}{\varepsilon_r}\right) D$

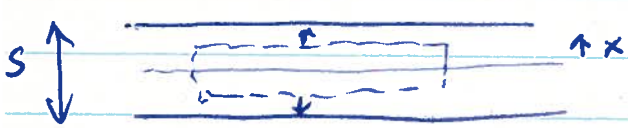
$$D = \frac{Q}{2A} = -2 \times 10^{-5} \ ; \quad E = \frac{D}{\varepsilon} = 7.06 \times 10^{-5}\ V/m$$

$$P = \left(1 - \frac{1}{\varepsilon_r}\right) D = -1.4 \times 10^{-5}\ C/m^2 = \sigma_b$$

c) - d) - e) $\qquad s = 2\ mm$

$$\frac{Q_f}{m^2} = \frac{\text{Free charge}}{m^2} = \frac{10^{-7}}{25 \times 10^{-4}} = 4 \times 10^{-5}\ \frac{C}{m^2}$$

Inside the charge $D = \frac{Qx}{sA} = 4 \times 10^{-5}\ \frac{x}{s}$



$$\oint D \cdot dA = Q_x = 2xA\rho$$

$$2DA = \ = 2xA\frac{Q}{sA}$$

$$D = \frac{Qx}{sA} = 4 \times 10^{-5}\ \frac{x}{s}$$

$$E = \frac{D}{\varepsilon_0 \cdot 3.2} = 1.4 \times 10^{6}\ \frac{x}{s}\ \frac{V}{m}$$

3) Cont'd

At 1 mm from center, $E = 1.4 \times 10^6 \cdot \frac{1}{2} = 7.06 \times 10^5$

$V = -E \cdot (6-1) \times 10^{-3} = -3.53$ kV,

setting $V = 0$ at surface of Lucite.

Inside charge

$$V = -3.53 \times 10^3 + 1.41 \times 10^6 \int_{10^{-6}}^{x} \frac{x}{s}\, dx$$

$$= -3.88 \times 10^3 + 3.53 \times 10^8 x^2$$

See GRAPH next page

Stored energy

$$2 \cdot \left[\frac{1}{2} \int_0^{t/2} \rho_f V \cdot \overset{A}{(25 \times 10^{-4})} dx \right]$$

$$= \int_0^{10^{-3}} (-2 \times 10^{-2})(-3.88 \times 10^3 + 3.53\, 10^8 x^2)(25 \times 10^{-4})\, dx$$

$$= 2 \times 10^{-4} \text{ J}$$

No explosion danger

8) Cont'd