UNIVERSITY OF WATERLOO
MIDTERM EXAMINATION
FALL TERM 2011

SOLUTIONS

Student Name (Print Legibly)	(FAMILY NAME)	(GIVEN NAME)
Signature		
Student ID Number		

COURSE NUMBER	SYDE 111
COURSE TITLE	Fundamental Engineering Math 1
DATE OF EXAM	Thursday, October 13th, 2011
TIME PERIOD	19:00 - 21:00
DURATION OF EXAM	120 minutes
NUMBER OF EXAM PAGES (Including this sheet)	7
INSTRUCTOR	**Mukto Akash**
EXAM TYPE	Closed Book
ADDITIONAL MATERIALS ALLOWED	NONE **(NO CALCULATORS)**

Notes:

1. Fill in your name, ID number, section, and sign the paper. *Don't write formulas on this page.*
2. Answer all questions in the space provided. If you run out of space, continue on the back of the preceding page, indicating where your work continues. The last page is for rough work.
3. Check that there are 7 sheets.
4. *Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.*

Marking Scheme:

Question	Mark	Out of
1		15
2		25
3		20
4		15
5		15
6		10
Total		100

1. Functions.

(a) Given $h(x) = \frac{1-x}{1+x}$, evaluate $h\left(\frac{1-x}{1+x}\right)$. **[3 marks]**

$$h\left(\frac{1-x}{1+x}\right) = \frac{1-\left(\frac{1-x}{1+x}\right)}{1+\left(\frac{1-x}{1+x}\right)} = \frac{\frac{1+x-(1-x)}{\cancel{1+x}}}{\frac{1+x+(1-x)}{\cancel{1+x}}}$$

$$= \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x.$$

(b) What is the domain of h? **[1 mark]**

$\{x \in \mathbb{R} : x \neq 1\}$

(c) What is the range of h? **[3 marks]**
(*Hint:* What happens as x approaches -1 from the left and from the right?)

First, note that $-1 = \frac{1-x}{1+x} \Rightarrow -(1+x) = 1-x \Rightarrow -1-x = 1-x$ has no solutions, so the range cannot contain -1. As, as $x \to -1^+$, we see that $\frac{1-x}{1+x} \to -\infty$ while as $x \to -1^-$, $\frac{1-x}{1+x} \to +\infty$, hence the range is given by the set $\{y \in \mathbb{R} : y \neq -1\}$

(d) Suggest an inverse function for h. **[2 marks]**

Since $h\left(\frac{1-x}{1+x}\right) = h \circ h(x) = x$,
therefore $h^{-1}(x) = h(x)$ (i.e., h is its own inverse!)

(e) Show that $h(x) = -h\left(\frac{1}{x}\right)$ and that $h(-x) = \frac{1}{h(x)}$ **[6 marks]**

$$h\left(1/x\right) = \frac{1-1/x}{1+1/x} = \frac{\frac{x-1}{\cancel{x}}}{\frac{x+1}{\cancel{x}}} = \frac{x-1}{x+1} = -\left(\frac{1-x}{1+x}\right) = -h(x)$$

$$\Rightarrow h(x) = -h\left(1/x\right). \quad \text{(Shown)}$$

Similarly,

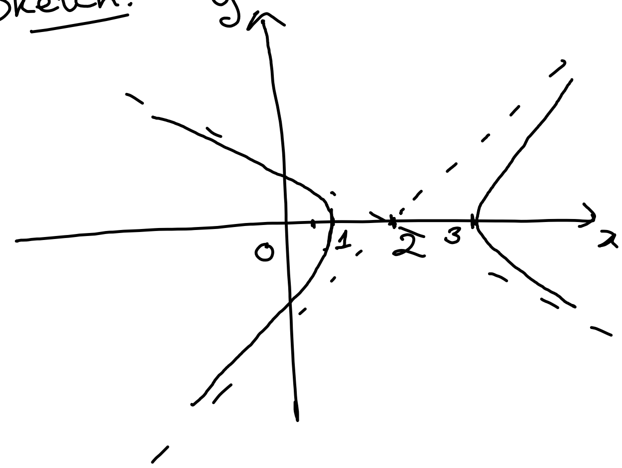
$$h(-x) = \frac{1-(-x)}{1+(-x)} = \frac{1+x}{1-x} = \frac{1}{\left(\frac{1-x}{1+x}\right)} = 1/h(x).$$

(Shown)

2. Rearrange the following equations in their standard forms and draw the corresponding conic sections on separate axes

(a) $x^2 - y^2 - 4x + 3 = 0$ **[10 marks]**

Sketch:



$$x^2 - y^2 - 4x + 3 = 0$$

$$\Rightarrow x^2 - 2\cdot\frac{4}{2}x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 - y^2 + 3 = 0$$

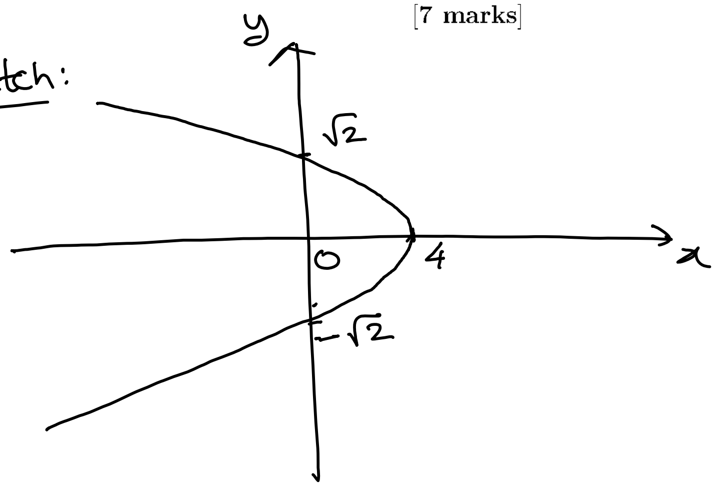
$$\Rightarrow (x-2)^2 - y^2 - 4 + 3 = 0$$

$$\Rightarrow (x-2)^2 - y^2 = 1$$

This is the standard form for a hyperbola.

(b) $x = 4 - 2y^2$ **[7 marks]**

Sketch:

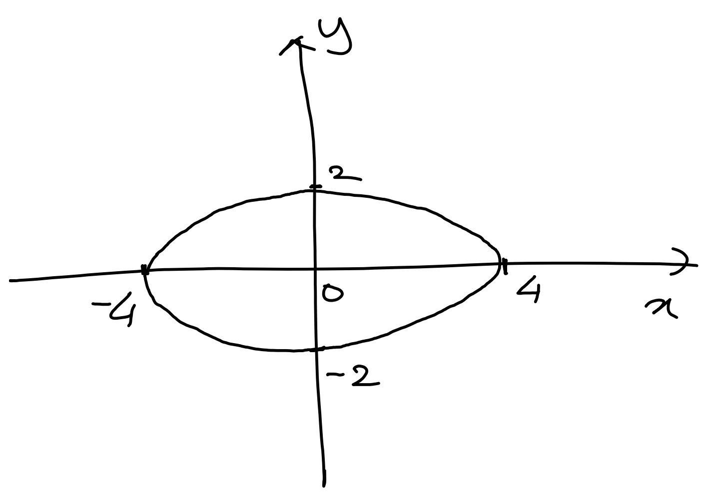


$$x = 4 - 2y^2$$

$$\Rightarrow x = -2y^2 + 4$$

This is a parabola

(c) $4x^2 + 16y^2 = 64$. **[8 marks]**



$$4x^2 + 16y^2 = 64$$

$$\Rightarrow \frac{4x^2}{64} + \frac{16y^2}{64} = 1$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

This is an ellipse.

3. Prove the given Trigonometric identities

(a) $\frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} = \tan\frac{x}{2}$ [**10 marks**]

Here you have to prove two identities:

First, we will prove: $\frac{1-\cos x}{\sin x} = \tan x/2$.

We know: $\cos(2x) = 1 - 2\sin^2 x \Rightarrow \cos(x) = 1 - 2\sin^2(x/2)$

and $\sin(2x) = 2\sin(x/2)\cos(x/2)$

Then: $\frac{1-\cos x}{\sin x} = \frac{1-[1-2\sin^2(x/2)]}{2\sin(x/2)\cos(x/2)} = \frac{\sin^2(x/2)}{\sin(x/2)\cos(\frac{x}{2})} = \tan(\frac{x}{2})$ (Shown).

Next, using $\cos(2x) = 2\cos^2 x - 1 \Rightarrow \cos(x) = 2\cos^2(x/2) - 1$

We get: $\frac{\sin x}{1+\cos x} = \frac{2\sin(x/2)\cos(x/2)}{1+(2\cos^2(x/2)-1)} = \frac{\sin(x/2)\cos(x/2)}{\cos^2(x/2)} = \tan(x/2)$ (Shown).

(b) $\sin(x)\sin(2x) + \cos(x)\cos(2x) = \cos(x)$ [**5 marks**]

$$\begin{aligned}
&\sin(x)\sin(2x) + \cos(x)\cos(2x) \\
&= \sin(x)(2\sin x\cos x) + \cos(x)(\cos^2 x - \sin^2 x) \\
&= 2\sin^2 x\cos x + \cos(x)\cos^2 x - \sin^2 x\cos x \\
&= \sin^2 x\cos x + \cos^2 x\cos x \\
&= [\sin^2 x + \cos^2 x]\cos x \\
&= \cos(x). \quad \text{(Proved)}
\end{aligned}$$

[Using: $\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x$]

[$\sin^2 x + \cos^2 x \equiv 1$]

(c) $\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = 2\sec^2 x$. [**5 marks**]

$$\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = \frac{(1+\sin x)+(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{2}{1-\sin^2 x} = \frac{2}{\cos^2 x}$$

[$\cos^2 x = 1 - \sin^2 x$]

$$= 2\sec^2 x \quad \text{(Shown)}$$

4. Consider the function $f(x) = 2 - e^{-x}$.

(a) By starting with an elementary function and applying appropriate transformations, sketch the given function, showing all the intercepts. Explain which transformations you have done to obtain the graph, and use dashed lines for the intermediate graphs. [**7 marks**]

Elementary fcn: e^x (purple)

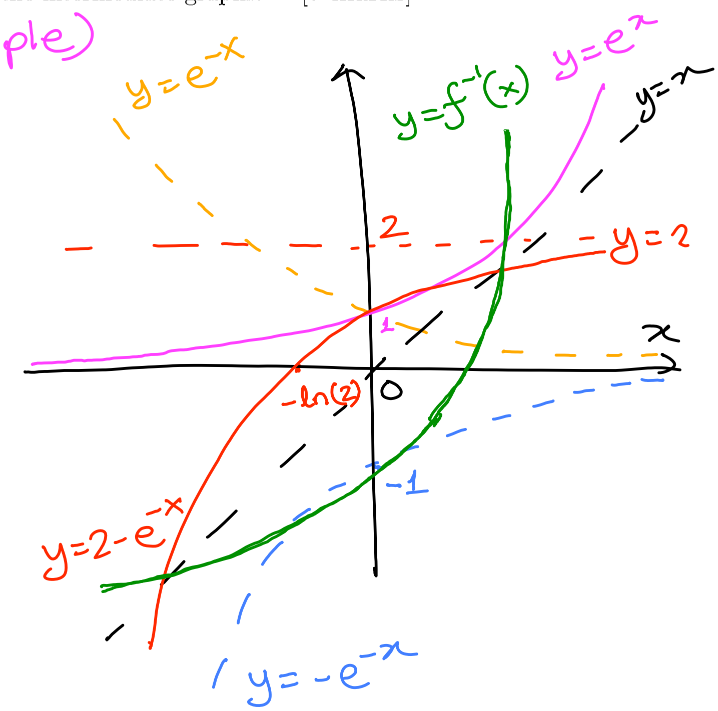
Transformations:

① $e^x \to e^{-x}$ (orange)
Reflect about y-axis

② $e^{-x} \to -e^{-x}$ (blue)
Reflect about x-axis

③ $-e^{-x} \to 2 - e^{-x}$ (red)
Shift 'up' by 2-units

Thus we get the final graph in red.



(b) Find a formula for $f^{-1}(x)$ and sketch $y = f^{-1}(x)$ on the axes above, being sure to label it clearly. [**8 marks**]

Given $f(x) = 2 - e^{-x}$.

Let $y = 2 - e^{-x} \Rightarrow y - 2 = -e^{-x}$

$\Rightarrow e^{-x} = 2 - y$

$\Rightarrow -x = \ln(2-y)$

$\Rightarrow x = -\ln(2-y)$. (Green)

$\Rightarrow f^{-1}(x) = -\ln(2-x)$

The sketch of $y = f^{-1}(x)$ is shown in green.

5. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0\left(1 - e^{-t/a}\right)$$

(The maximum charge capacity is Q_0 and t is measured in seconds).

(a) Find the inverse of this function and explain its meaning. [**8 marks**]

Given $Q = Q_0(1 - e^{-t/a})$

$\Rightarrow Q/Q_0 = 1 - e^{-t/a}$

$\Rightarrow e^{-t/a} = 1 - Q/Q_0$

$\Rightarrow -t/a = \ln(1 - Q/Q_0)$

$\Rightarrow t = -a\ln(1 - Q/Q_0)$

This gives us the time t (in seconds) required to charge the capacitor to charge Q.

(b) How long does it take to recharge the capacitor to 90% of capacity if a = 2? Simplify your answer and leave it in exact form (no decimals). [**7 marks**]

Since $a = 2$, then $t = -2\ln(1 - Q/Q_0)$

At 90% of the capacity, we have $Q = 0.9Q_0$

$\Rightarrow t = -2\ln\left(1 - \frac{0.9Q_0}{Q_0}\right) = -2\ln(1 - 0.9)$

$= -2\ln(0.1) = 2\ln(10) = \ln(100)$ s.

6 (a). What does $\lim_{x\to a^+} f(x) = L$ mean for a function f, and real numbers a and L? Be sure to define the concept represented by this symbol. [**4 marks**]

$\lim_{x\to a^+} f(x) = L$ means that the values of $f(x)$ get closer and closer to L as x approaches a, and $x > a$.

(The symbol represents that the Right Hand limit of $f(x)$ as x approaches 'a' is L, but that's not the defn).

(b) Find the following limits using the rules for computing limits [**6 marks**]

(i) $\lim_{x\to 2} \frac{x^3+x}{x+2}$

Since $\lim_{x\to 2} x+2 = 2+2 = 4 \neq 0$, therefore

$$\lim_{x\to 2} \frac{x^3+x}{x+2} = \frac{\lim_{x\to 2}(x^3+x)}{\lim_{x\to 2}(x+2)} \text{ [By the limit laws] } = \frac{2^3+2}{2+2} = \frac{10}{4} = 5/2$$

(ii) $\lim_{x\to 1^+} \sec(\ln(x))$.

$\lim_{x\to 1^+} \sec(\ln(x)) = \lim_{x\to 1^+} \frac{1}{\cos(\ln(x))}$. Since $\lim_{x\to 1^+} \ln(x) = \ln(1) = 0$ is in the domain of $\cos(x)$ and $\cos(0) = 1 \neq 0$, we get

$$= \frac{1}{\cos\left(\lim_{x\to 1^+} \ln(x)\right)} = \frac{1}{\cos(0)} = 1.$$

You may use the space below for rough work (in which case you may tear off this page), or to continue any other question that you have ran out of space answering. In this case, be sure to indicate clearly, in the original location, that the work continues here.