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MAT 1330
Calculus for the Life Sciences I
Final Exam-AID
Review Package

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Discrete-Time Dynamical Systems (DTDS)

- **General solutions** of DTDS with a linear updating function $x_{t+1} = rx_t + s$:

When $s = 0$ $x_{t+1} = rx_t$	When $r = 1$ $x_{t+1} = x_t + s$	When $s \neq 0$ and $r \neq 1$ $x_{t+1} = rx_t + s$
$x_t = r^t x_0$ If r>1, pop. increases If r = 1, pop. remains constant If r<1, pop. decreases	$x_t = x_o + st$	$x_t = ar^t + b$

- A point x^* is an **equilibrium** of $x_{t+1} = f(x_t)$ if $f(x^*) = x^*$

1. Consider the DTDS

$$x_{t+1} = \frac{1 + x_t}{1 + x_t^2}, \quad t = 0,1,2,\ldots$$

   a. State the updating function of this DTDS.
   b. Find the only positive steady state x^*.
   c. Using the derivative test, is the steady state stable or unstable?
   d. Starting from $x_0 = 4$, calculate x_1, x_2.

2. A new rule has been approved at the University that stipulates that precisely ¼ of all students must leave at the end of each summer (and the remaining students must stay) and exactly 11,000 new students will be added at the same time.
   a. What is the discrete-time dynamical system that gives the number of students at the university in a given year?
   b. The number of students at the university is now 34,815. How many students will there be two years later (round your answer)?
   c. Write down the updating function of the dynamical system.
   d. Find all equilibrium points of the dynamical system.
   e. Find a formula for the solution to the dynamical system for initial condition S_0=65,000.
   f. Draw the solution of the dynamical system with S_0=65,000. (4 points are enough)
   g. Draw the cobweb diagram of the dynamical system with S_0=65,000.
   h. Determine the stability of the equilibrium point using the cobweb diagram.

3. Consider the following discrete-time model for the population of a species:

$$N_{t+1} = \frac{rN_t}{1-N_t}$$

For what values of r (which is a constant) does the model have a positive equilibrium?

4. A population of rabbits is growing at a rate of 12% per year. Write down the discrete dynamical system that describes the evolution of the rabbit population. If the population is initially 1000, how many years will it take for the population to exceed 100,000?

5. The concentration of a drug in the body of a patient is reduced by 60% per day. The daily dose of this drug is d. The DTDS modeling the concentration x_t of the drug in the body on day t is $x_{t+1} = 0.4x_t + d$
   a. What is the updating function of the DTDS?
   b. What is the equilibrium point of the DTDS?
   c. Assume the daily dose is $d = 4$. Give the solution formula for the DTDS with general initial condition x_0.
   d. For a patient with an initial concentration of $x_0 = 3$ and a daily dose of $d = 4$, what is the concentration on day 3?
   e. Graph the updating function for $d = 4$ and draw the cobweb diagram of the DTDS, starting from $x_0 = 3$ for at least 4 steps.
   f. Is the equilibrium point stable or unstable?
   g. Suppose that the doctors recommend a concentration of 15 in the long run. How do they have to choose the daily dose d to obtain this value?

6. Consider the discrete-time dynamical system

$$M_{t+1} = -0.5M_t + 1$$

   a. Find the updating function of the DTDS.
   b. Find the equilibrium point of the DTDS.
   c. Give the general solution formula for the DTDS.
   d. Calculate M_8 if $M_0 = 0$
   e. Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.
   f. Is the equilibrium point stable or unstable?

Limits

1. Determine if the following limits exist. If the limit exists, compute the limit.
   a. $\lim_{x\to 1} \frac{x^2+2x-3}{x^2-x}$
   b. $\lim_{x\to 2} \frac{x^2+x-6}{x^2-2x+4}$
   c. $\lim_{x\to 0} \frac{\sin(4x)}{3x}$
   d. $\lim_{h\to 0} \frac{(3+h)^2-3^2}{\sqrt{3+h}-\sqrt{3}}$
   e. $\lim_{x\to 0} \frac{x}{1-\sqrt{1+x}}$
   f. $\lim_{x\to\infty} \frac{\pi x^4+2x^3+\pi}{x^4+x}$
   g. $\lim_{x\to 0^+} x\ln x$
   h. $\lim_{x\to\infty} \frac{x^3}{e^x}$
   i. $\lim_{x\to 0} \frac{a^x-b^x}{\sin x}$
2. Let $f(x)=\frac{1-x^2}{|x-1|}+|x+1|$.
   a. Find $\lim_{x\to -1^+} f(x)$.
   b. Find $\lim_{x\to -1^-} f(x)$.
   c. Does $\lim_{x\to -1} f(x)$ exist? Justify your answer.
3. Consider the function $f(x)=\frac{1+3e^{-x}}{1-e^{-x}}$.
   a. Find $\lim_{x\to\infty} f(x)$
   b. Find $\lim_{x\to -\infty} f(x)$.
4. Consider the function
   $f(x)=\{a\sin(\frac{\pi}{2}x+b) \text{ if } x\le 0\}, \{x^2-a \text{ if } 0<x\le 1\}, \{b\cos(\pi x)+a \text{ if } x>1\}$.
   Find the values *a* and *b* for which the function is continuous everywhere.

5. Consider the function $f(x) = \{\frac{a}{(\sin(x))^2 + 2} \text{ if } x < \frac{\pi}{2}\}, \{\frac{bx+1}{x+1} \text{ if } x \geq \frac{\pi}{2}\}$
    a. Find the conditions for a and b such that the function is continuous at $\pi/2$.
    b. Find a and b so that the function is continuous and has the horizontal asymptote $y = 2$ as $x \to \infty$.

Derivatives

$y = f(x)$	$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$
$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = \sin x$	$f'(x) = \cos x$
$f(x) = \cos x$	$f'(x) = -\sin x$
$f(x) = \tan x$	$f'(x) = \sec^2 x$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \ln x$	$f'(x) = \frac{1}{x}$
$f(x) = \sec x$	$f'(x) = \sec x \tan x$
$f(x) = \csc x$	$f'(x) = -\csc x \cot x$
$f(x) = \cot x$	$f'(x) = -\csc^2 x$
$f(x) = a^x$	$f'(x) = a^x(\ln a)$
$f(x) = \log_a x$	$f'(x) = \frac{1}{x \ln a}$
$f(x) = \arcsin x$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$f(x) = \arccos x$	$f'(x) = -\frac{1}{\sqrt{1-x^2}}$
$f(x) = \arctan x$	$f'(x) = \frac{1}{1+x^2}$

1. Use the definition of the derivative (first principles) to calculate the derivative of the following:
    a. $f(x) = 5x + 4$
    b. $f(x) = \frac{1}{2x+1}$
    c. $f(x) = \sqrt{x^2 - 3}$

2. Find the derivatives of the following with respect to x:
    a. $f(x) = \frac{1}{\sqrt{4}} e^{-(x-4)^2}$
    b. $f(x) = \ln\left(\frac{1}{x^4 + 3}\right)$
    c. $f(x) = e^{2ax} \tan(bx) + \sin^2 x$
    d. $f(x) = \ln\left(6e^{x^3}(x^3 - 3)^2\right)$
    e. $f(x) = e^{(10x+11)} \ln(x^{2010} + 1)$
    f. $f(x) = \cos^2(\sqrt{x})$
3. Consider the function $f(x) = \frac{1-x}{x-3}$
    a. Find the domain of f.
    b. Find the limits of f as x approaches $\pm\infty$
    c. Are there points where f is not continuous? If yes, find the left and right limit in each case.
    d. Find the intervals where f is increasing and decreasing. Are there critical points?
    e. Find the intervals where f is concave up or concave down.
    f. Draw the graph of f.
4. Consider the function $f(x) = \frac{1}{x^2} - \frac{1}{2x^3}$
    a. Find the domain of f.
    b. Find the x-intercepts of f.
    c. Find the derivative of f.
    d. Find the critical points of f.
    e. Find the second derivative of f.
    f. Find the point(s) of inflection.
    g. Find the $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$
    h. Find the $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$
    i. Sketch the graph of f for $x \in [-5,5]$.
5. Consider the function $f(x) = e^{-x} - x$
    a. Explain why this function has a zero in the interval [0,1].
    b. Calculate the zero to 3 decimal places.
6. Find the Taylor polynomial of degree 4 for $f(x) = \sin(2x) + x^2$ with base point *a = 0*. (*x* in radians!)

Integrals

Standard integrals	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$ $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$ $\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ $\int e^x\,dx = e^x + C$ $\int a^x dx = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C, \lvert x\rvert < a$
Integration by parts	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ **OR** $\int u\,dv = uv - \int v\,du$

1. Evaluate the following indefinite integrals:
   a. $\int (x^4 + \cos x)\,dx$
   b. $\int \frac{e^{3x} + 4}{e^{3x}}\,dx$
   c. $\int 16x^3 \ln(8x)\,dx$
   d. $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx$
   e. $\int \ln x\,dx$
   f. $\int e^x \cos x\,dx$

2. An athlete starts a marathon with a speed of 14 km/h. Due to an existing ankle injury she is forced to slow down and her speed decreases at a constant rate according to the differential equation $v'(t) = -3, t \geq 0$ with $v(0) = 15$. Answer the following questions:
   a. Find the equation for the speed in km/h as a function of time in hours.
   b. Solve the pure-time differential equation $\frac{dp}{dt} = v(t), p(0) = 0$ for the location p in km.
   c. How long will it take the athlete to complete the marathon (42 km)?

Discrete-Time Dynamical Systems (DTDS) - Solutions

1. Consider the DTDS

$$x_{t+1} = \frac{1+x_t}{1+x_t^2}, \quad t = 0,1,2,\ldots$$

   a. State the updating function of this DTDS.

$$f(x_t) = \frac{1+x_t}{1+x_t^2}$$

   b. Find the only positive steady state x^*.

$$x^* = \frac{1+x^*}{1+x^{*2}}$$
$$x^* + x^{*3} = 1 + x^*$$
$$x^{*3} = 1$$
$$x^* = 1$$

   c. Using the derivative test, is the steady state stable or unstable?

$$f(x_t) = \frac{1+x_t}{1+x_t^2}$$

$$f'(x_t) = \frac{(1+x_t^2)-(1+x_t)(2x_t)}{(1+x_t^2)^2} = \frac{1+x_t^2-2x_t-2x_t^2}{(1+x_t^2)^2} = -\frac{x_t^2+2x_t-1}{(1+x_t^2)^2}$$

$$f'(1) = -\frac{(1)^2+2(1)-1}{(1+(1)^2)^2} = -\frac{2}{4}$$

$$\left|f'(1)\right| < 1$$
$$\frac{1}{2} < 1$$

Therefore the steady state is **stable**

   d. Starting from $x_0 = 4$, calculate x_1, x_2.

$$f(x_t) = \frac{1+x_t}{1+x_t^2}$$
$$x_1 = \frac{1+(4)}{1+(4)^2}$$
$$x_1 = \frac{5}{17}$$

$$x_2 = \frac{1+\left(\frac{5}{17}\right)}{1+\left(\frac{5}{17}\right)^2} = 1.19$$

2. A new rule has been approved at the University that stipulates that precisely ¼ of all students must leave at the end of each summer (and the remaining students must stay) and exactly 11,000 new students will be added at the same time.
    a. What is the discrete-time dynamical system that gives the number of students at the university in a given year?

$$S_{t+1} = \frac{3}{4} S_t + 11{,}000$$

    b. The number of students at the university is now 34,815. How many students will there be two years later (round your answer)?

$$S_1 = \frac{3}{4}(34{,}815) + 11{,}000$$
$$S_1 = 37{,}111.25$$
$$S_2 = \frac{3}{4}(37{,}111.25) + 11{,}000$$
$$S_2 = 38{,}833$$

    c. Write down the updating function of the dynamical system.

$$f\left(S_t\right) = \frac{3}{4} S_t + 11{,}000$$

    d. Find all equilibrium points of the dynamical system.

$$S^* = \frac{3}{4} S^* + 11{,}000$$
$$\frac{1}{4} S^* = 11{,}000$$
$$S^* = 44{,}000$$

    e. Find a formula for the solution to the dynamical system for initial condition S_0=65,000.

$$S_{t+1} = \frac{3}{4} S_t + 11{,}000$$

$$\text{Let } S_t = a\left(\frac{3}{4}\right)^t + b$$

$$S_0 = 65{,}000$$
$$S_1 = \frac{3}{4}(65{,}000) + 11{,}000$$
$$S_1 = 59{,}750$$

$$S_0 = a\left(\frac{3}{4}\right)^0 + b$$

$$65{,}000 = a + b$$

$$b = 65{,}000 - a$$

$$S_1 = a\left(\frac{3}{4}\right)^1 + b$$

$$59{,}750 = \frac{3}{4}a + b$$

$$59{,}750 = \frac{3}{4}a + (65{,}000 - a)$$

$$-5{,}250 = -\frac{1}{4}a$$

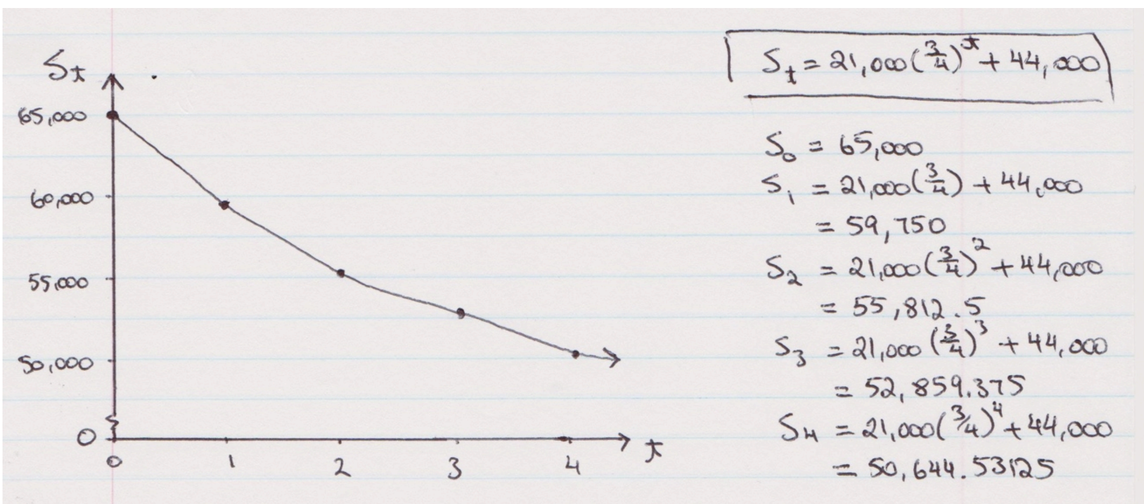
$$21{,}000 = a$$

$$b = 65{,}000 - (21{,}000)$$

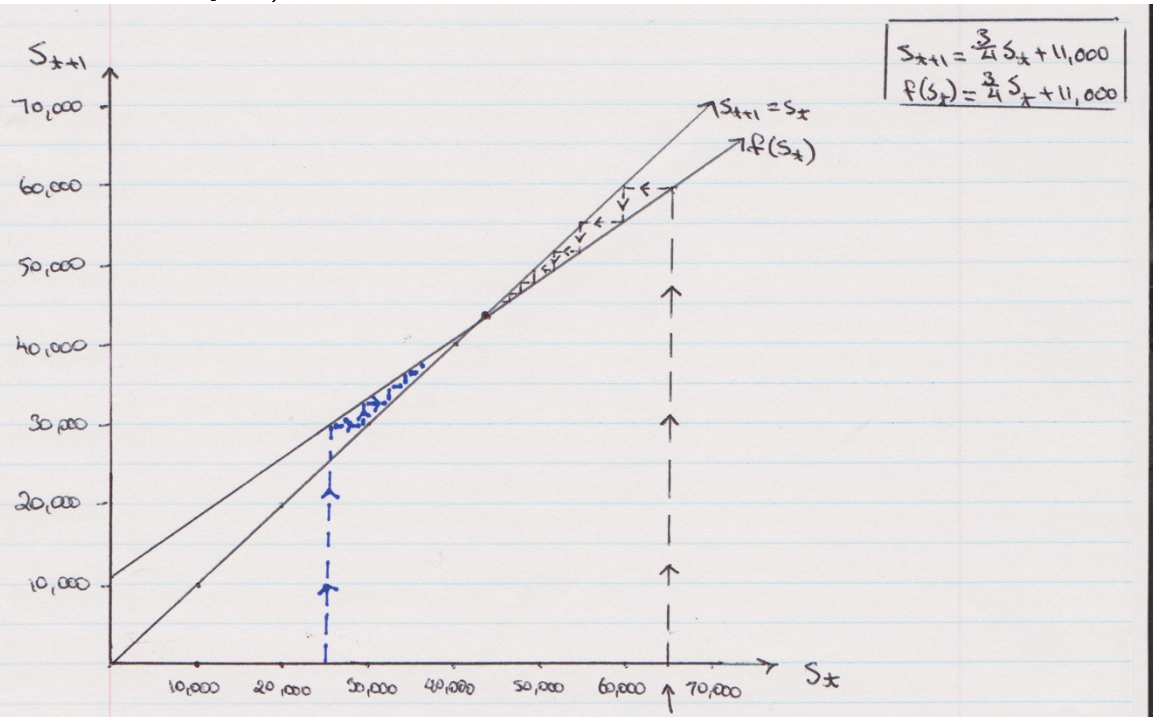
$$b = 44{,}000$$

$$\therefore S_t = 21{,}000\left(\frac{3}{4}\right)^t + 44{,}000$$

f. Draw the solution of the dynamical system with S_0=65,000. (4 points are enough)



g. Draw the cobweb diagram of the dynamical system with S_0=65,000.



h. Determine the stability of the equilibrium point using the cobweb diagram.

The equilibrium point S^*=44,000 is stable because solutions near the equilibrium do approach the equilibrium point.

3. Consider the following discrete-time model for the population of a species:

$$N_{t+1} = \frac{rN_t}{1-N_t}$$

For what values of r (a constant) does the model have a positive equilibrium?

$$N^* = \frac{rN^*}{1-N^*}$$

$$N^*(1-N^*) - rN^* = 0$$

$$N^*(-N^* - (r-1)) = 0$$

$$\therefore N^* = 0, N^* = 1-r$$

$1-r$ is positive when $r<1$. Therefore $r<1$ is the answer.

4. A population of rabbits is growing at a rate of 12% per year. Write down the discrete dynamical system that describes the evolution of the rabbit population. If the population is initially 1000, how many years will it take for the population to exceed 100,000?

$$x_{t+1} = 1.12x_t$$
$$x_t = r^t x_0$$
$$x_t = (1.12)^t(1000)$$

$$100000 < (1.12)^t 1000$$
$$100 < (1.12)^t$$
$$\log 100 < \log(1.12)^t$$
$$\log 100 < t\log(1.12)$$
$$\frac{\log 100}{\log 1.12} < t$$

$\therefore$ It will take over $\frac{\log 100}{\log 1.12}$ years or 40.635 years or 41 years

5. The concentration of a drug in the body of a patient is reduced by 60% per day. The daily dose of this drug is *d*. The DTDS modeling the concentration x_t of the drug in the body on day *t* is $x_{t+1} = 0.4x_t + d$

   i. What is the updating function of the DTDS?

$$f(x) = \frac{2}{5}x + d$$

   j. What is the equilibrium point of the DTDS?

$$x^* = \frac{2}{5}x^* + d$$
$$\frac{3}{5}x^* = d$$
$$x^* = \frac{5}{3}d$$

   k. Assume the daily dose is *d* = *4*. Give the solution formula for the DTDS with general initial condition x_0.

$$x_{t+1} = \frac{2}{5}x_t + 4$$

Let $x_t = a\left(\frac{2}{5}\right)^t + b$

$$x_0 = a\left(\frac{2}{5}\right)^0 + b$$ ← Using general solution

$$x_0 = a + b$$

$$x_1 = \frac{2}{5}x_0 + 4$$ ← Using updating function

$$x_1 = a\left(\frac{2}{5}\right)^1 + b$$

$$\frac{2}{5}x_0 + 4 = \frac{2}{5}a + b$$

$$\frac{2}{5}x_0 + 4 = \frac{2}{5}(x_0 - b) + b$$

$$\frac{2}{5}x_0 + 4 = \frac{2}{5}x_0 - \frac{2}{5}b + b$$

$$4 = \frac{3}{5}b$$

$$\frac{20}{3} = b$$

$$x_0 = a + b$$

$$a = x_0 - b$$

$$a = x_0 - \frac{20}{3}$$

Therefore $x_t = ar^t + b$

$$x_t = \left(x_0 - \frac{20}{3}\right)\left(\frac{2}{5}\right)^t + \frac{20}{3}$$

l. For a patient with an initial concentration of x_0 = 3 and a daily dose of d = 4, what is the concentration on day 3?

$$x_t = \left(x_0 - \frac{20}{3}\right)\left(\frac{2}{5}\right)^t + \frac{20}{3}$$

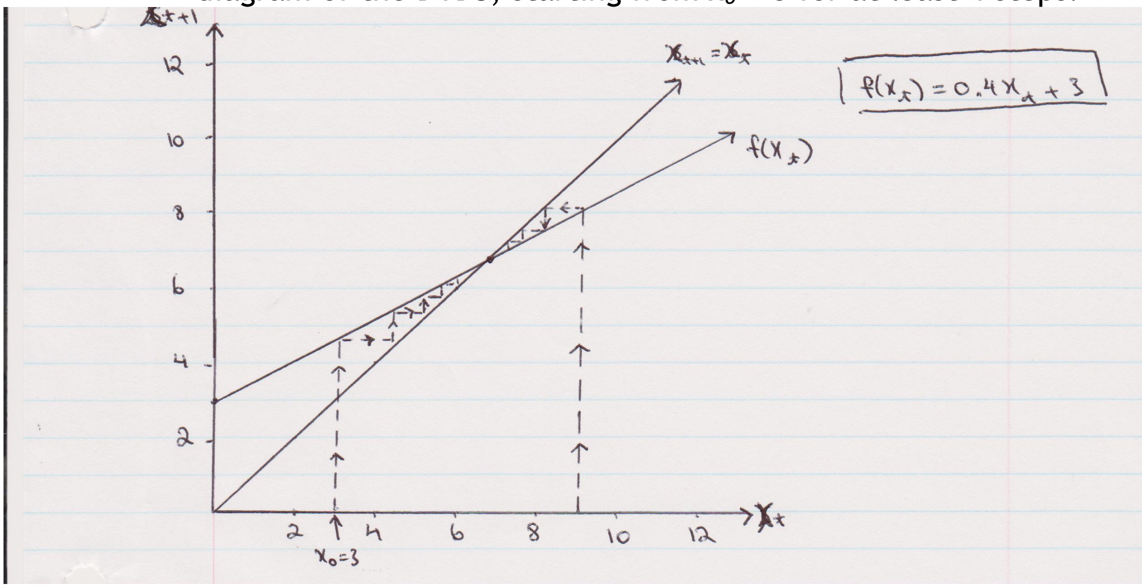
$$x_3 = \left((3) - \frac{20}{3}\right)\left(\frac{2}{5}\right)^{(3)} + \frac{20}{3}$$

$$x_3 = \left(-\frac{11}{3}\right)\left(\frac{2}{5}\right)^{(3)} + \frac{20}{3}$$

$$x_3 = -\frac{88}{375} + \frac{20}{3}$$

$$x_3 = \frac{2412}{375} = 6.432$$

m. Graph the updating function for d = 4 and draw the cobweb diagram of the DTDS, starting from x_0 = 3 for at least 4 steps.



n. Is the equilibrium point stable or unstable?

The equilibrium point is stable

o. Suppose that the doctors recommend a concentration of 15 in the long run. How do they have to choose the daily dose d to obtain this value?

$$x^* = \frac{5}{3}d$$

$$15 = \frac{5}{3}d$$

$$9 = d$$

6. Consider the discrete-time dynamical system

$$M_{t+1} = -0.5M_t + 1$$

g. Find the updating function of the DTDS.

$$f(M) = -\frac{1}{2}M + 1$$

h. Find the equilibrium point of the DTDS.

$$M^* = -\frac{1}{2}M^* + 1$$

$$\frac{3}{2}M^* = 1$$

$$M^* = \frac{2}{3}$$

i. Give the general solution formula for the DTDS.

$$M_0 = a\left(-\frac{1}{2}\right)^0 + b$$ ← Using general solution

$$M_0 = a + b$$

$$M_1 = -\frac{1}{2}M_0 + 1$$ ← Using updating function

$$M_1 = a\left(-\frac{1}{2}\right)^1 + b$$

$$-\frac{1}{2}M_0 + 1 = a\left(-\frac{1}{2}\right) + b$$

$$-\frac{1}{2}M_0 + 1 = (M_0 - b)\left(-\frac{1}{2}\right) + b$$

$$a = M_0 - b$$

$$a = M_0 - \frac{2}{3}$$

$$-\frac{1}{2}M_0 + 1 = -\frac{1}{2}M_0 + \frac{1}{2}b + b$$

$$1 = \frac{3}{2}b$$

$$\frac{2}{3} = b$$

Therefore $M_t = ar^t + b$

$$M_t = \left(M_0 - \frac{2}{3}\right)\left(-\frac{1}{2}\right)^t + \frac{2}{3}$$

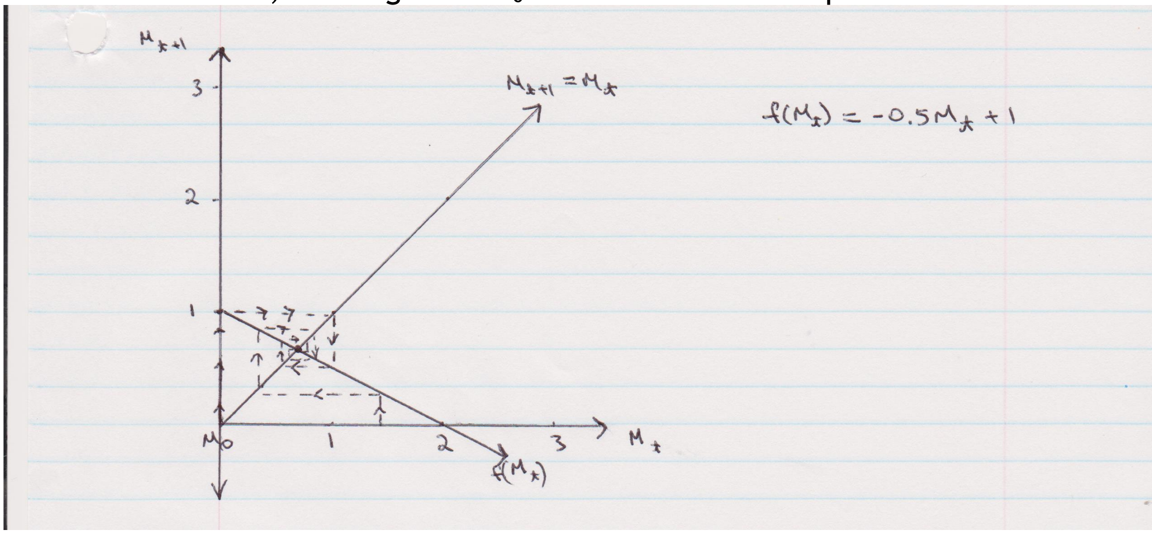
j. Calculate M_8 if $M_0 = 0$

$$M_t = \left(M_0 - \frac{2}{3}\right)\left(-\frac{1}{2}\right)^t + \frac{2}{3}$$

$$M_8 = \left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)^8 + \frac{2}{3}$$

$$M_8 = \frac{2}{768} + \frac{2}{3} = \frac{514}{768} = 0.669$$

k. Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.



l. Is the equilibrium point stable or unstable?
The equilibrium point is stable.

Limits Solutions

1. Evaluate the following limits:

a. $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-x}$

If you substitute $x = 1$ into the function you get 0/0, which is indeterminate.

$$\therefore \lim_{x\to 1}\frac{x^2+2x-3}{x^2-x} = \lim_{x\to 1}\frac{(x+3)(x-1)}{x(x-1)} = \lim_{x\to 1}\frac{x+3}{x} = 4$$

b. $\lim_{x\to 2}\frac{x^2+x-6}{x^2-2x+4}$

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-2x+4} = \frac{2^2+2-6}{2^2-2(2)+4} = \frac{0}{4} = 0$$

c. $\lim_{x\to 0}\frac{\sin(4x)}{3x}$

If you substitute $x = 0$ into the function, you get 0/0, which is indeterminate.

Recall that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$

$$\therefore \lim_{x\to 0}\frac{\sin(4x)}{3x} = \lim_{x\to 0}\frac{\sin(4x)}{3x}\left(\frac{4x}{4x}\right) = \lim_{x\to 0}\frac{4x}{3x}\frac{\sin(4x)}{4x} = \frac{4}{3}(1) = \frac{4}{3}$$

d. $\lim_{h\to 0}\frac{(3+h)^2-3^2}{\sqrt{3+h}-\sqrt{3}}$

If you substitute h = 0 into the function, you get 0/0, which is indeterminate.

$$\lim_{h\to 0}\frac{(3+h)^2-3^2}{\sqrt{3+h}-\sqrt{3}}=\lim_{h\to 0}\frac{(3+h)^2-3^2}{\sqrt{3+h}-\sqrt{3}}\left(\frac{\sqrt{3+h}+\sqrt{3}}{\sqrt{3+h}+\sqrt{3}}\right)=\lim_{h\to 0}\frac{\left((3+h)^2-3^2\right)\left(\sqrt{3+h}+\sqrt{3}\right)}{3+h-3}=\text{"}\frac{2\sqrt{3}}{0}\text{"}=+\infty$$

e. $\lim_{x\to 0}\frac{x}{1-\sqrt{1+x}}$ L'Hopitals Rule

If you substitute x = 0 into the function, you get 0/0, which is indeterminate.

$$\lim_{x\to 0}\frac{x}{1-\sqrt{1+x}}=\lim_{x\to 0}\frac{1}{\frac{1}{2}(1+x)^{-\frac{1}{2}}}=2\lim_{x\to 0}\sqrt{1+x}=2$$

f. $\lim_{x\to\infty}\frac{\pi x^4+2x^3+\pi}{x^4+x}$

Recall that $\lim_{n\to\infty}\frac{1}{x^n}=0$, $n\geq 0$

$$\lim_{x\to\infty}\frac{\pi x^4+2x^3+\pi}{x^4+x}=\lim_{x\to\infty}\frac{\frac{\pi x^4}{x^4}+\frac{2x^3}{x^4}+\frac{\pi}{x^4}}{\frac{x^4}{x^4}+\frac{x}{x^4}}=\lim_{x\to\infty}\frac{\pi+\frac{2}{x}+\frac{\pi}{x^4}}{1+\frac{1}{x^3}}=\frac{\pi+0+0}{1+0}=\pi$$

g. $\lim_{x\to 0^+}x\ln x$

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}=-\lim_{x\to 0^+}x=0$$

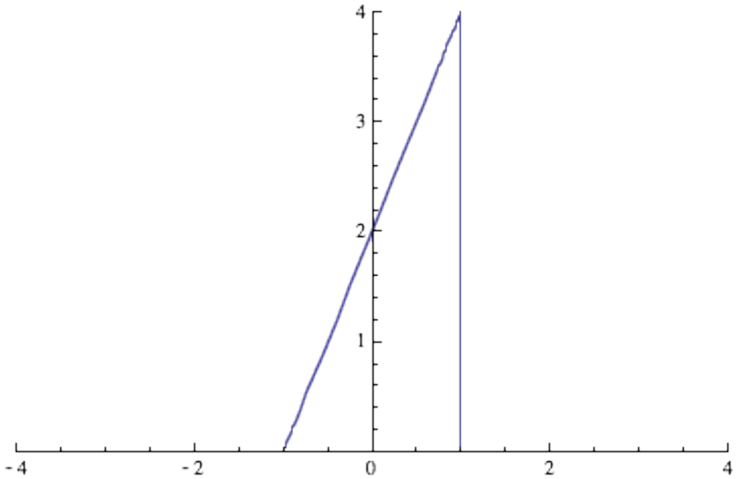
h. $\lim_{x\to\infty}\frac{x^3}{e^x}$

$$\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=0$$

i. $\lim_{x\to 0}\frac{a^x-b^x}{\sin x}$

$$\lim_{x\to 0}\frac{a^x-b^x}{\sin x}=\lim_{x\to 0}\frac{a^x\ln a-b^x\ln b}{\cos x}=\ln a-\ln b=\ln\left(\frac{a}{b}\right)$$

2. Let $f(x) = \dfrac{1-x^2}{|x-1|} + |x+1|$.



Graph of f(x)

a. Find $\lim_{x \to -1^+} f(x)$.

As $x \to -1^+$, $|x-1| = x-1$ and $|x+1| = x+1$

$$\begin{aligned}
\lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+}\left(\frac{1-x^2}{|x-1|} + |x+1|\right) \\
&= \lim_{x \to -1^+}\left(\frac{1-x^2}{x-1} + x + 1\right) \\
&= \lim_{x \to -1^+}\left(\frac{(1+x)(1-x)}{x-1} + x + 1\right) \\
&= \lim_{x \to -1^+}\left(\frac{(1+x)(1-x)}{-(1-x)} + x + 1\right) \\
&= \lim_{x \to -1^+}\left(-(1+x) + x + 1\right) \\
&= \lim_{x \to -1^+}\left(-1 - x + x + 1\right) \\
&= 0
\end{aligned}$$

b. Find $\lim_{x\to -1^-} f(x)$.

As $x \to -1^-$, $|x-1| = x-1$ and $|x+1| = -(x+1)$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\left(\frac{1-x^2}{|x-1|} + |x+1|\right)$$
$$= \lim_{x\to -1^-}\left(\frac{1-x^2}{x-1} - (x+1)\right)$$
$$= \lim_{x\to -1^-}\left(\frac{(1+x)(1-x)}{x-1} - x - 1\right)$$
$$= \lim_{x\to -1^-}\left(\frac{(1+x)(1-x)}{-(1-x)} - x - 1\right)$$
$$= \lim_{x\to -1^-}\left(-(1+x) - x - 1\right)$$
$$= \lim_{x\to -1^-}\left(-1 - x - x - 1\right)$$
$$= -2 + 2$$
$$= 0$$

c. Does $\lim_{x\to -1} f(x)$ exist? Justify your answer.

$\lim_{x\to -1} f(x)$ does exist because $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 0$

3. Consider the function $f(x) = e^{-x} - x$

a. Explain why this function has a zero in the interval [0,1].

Since *f(0)* > *0* and *f(1)* < *0* then by the intermediate value theorem there exists a value x^* with $0 < x^* < 1$ such that $f(x^*) = 0$

b. Calculate the zero to 3 decimal places.

$$f(x) = e^{-x} - x$$
$$f'(x) = -e^{-x} - 1$$

$$x_0 = 0.5, x_1 = 0.5 - \frac{e^{-0.5} - 0.5}{-e^{-0.5} - 1} = 0.56631$$
$$x_1 = 0.56631, x_2 = 0.56631 - \frac{e^{-0.56631} - 0.56631}{-e^{-0.56631} - 1} = 0.56714$$
$$x_2 = 0.56714, x_3 = 0.56714 - \frac{e^{-0.56714} - 0.56714}{-e^{-0.56714} - 1} = 0.56714$$

4. Consider the function $f(x)=\frac{1+3e^{-x}}{1-e^{-x}}$.

   a. Find $\lim_{x\to\infty} f(x)$

$$\text{Recall that } \lim_{x\to\infty}\frac{1}{e^x}=0$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{1+3e^{-x}}{1-e^{-x}}\right)=\lim_{x\to\infty}\left(\frac{1+\frac{3}{e^x}}{1-\frac{1}{e^x}}\right)=\left(\frac{1+0}{1-0}\right)=1$$

   b. Find $\lim_{x\to-\infty} f(x)$.

$$\text{Recall that } \lim_{x\to-\infty} e^x=\lim_{x\to\infty}\frac{1}{e^x}=0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\left(\frac{1+3e^{-x}}{1-e^{-x}}\right)=\lim_{x\to-\infty}\left(\frac{\frac{e^x+3}{e^x}}{\frac{e^x-1}{e^x}}\right)=\lim_{x\to-\infty}\left(\frac{e^{-x}(e^x+3)}{e^{-x}(e^x-1)}\right)=\lim_{x\to-\infty}\left(\frac{e^x+3}{e^x-1}\right)=\left(\frac{0+3}{0-1}\right)=-3$$

5. Consider the function

   $f(x)=\{a\sin(\frac{\pi}{2}x+b)$ if $x\le 0\}$, $\{x^2-a$ if $0<x\le 1\}$, $\{b\cos(\pi x)+a$ if $x>1\}$.

   Find the values *a* and *b* for which the function is continuous everywhere.

In order to be continuous everywhere, we must examine the limits at 0 and 1 such that $\lim_{x\to0^-} f(x)=\lim_{x\to0^+} f(x)$ and $\lim_{x\to1^-} f(x)=\lim_{x\to1^+} f(x)$

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^+} f(x)$$

$$\lim_{x\to0^-}\left(a\sin\left(\frac{\pi}{2}\right)x+b\right)=\lim_{x\to0^+}\left(x^2-a\right)$$

$$b=-a$$

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^+} f(x)$$

$$\lim_{x\to1^-}\left(x^2-a\right)=\lim_{x\to1^+}\left(b\cos\left(\pi x\right)\right)+a$$

$$1-a=-b+a$$

$$1=-(-a)+2a$$

$$\frac{1}{3}=a$$

$$\therefore b=-\frac{1}{3}$$

$$\therefore a=\frac{1}{3}, b=-\frac{1}{3} \text{ for the function to be continuous everywhere.}$$

6. Consider the function $f(x)=\{\frac{a}{(\sin(x))^2+2} \text{ if } x<\frac{\pi}{2}\}, \{\frac{bx+1}{x+1} \text{ if } x\geq\frac{\pi}{2}\}$
   a. Find the conditions for *a* and *b* such that the function is continuous at $\pi/2$.

In order for the function to be continuous at $\pi/2$, we must find *a* and *b* such that $\lim_{x\to\pi/2^-} f(x)=\lim_{x\to\pi/2^+} f(x)$.

$$\lim_{x\to\pi/2^-} f(x)=\lim_{x\to\pi/2^+} f(x)$$

$$\lim_{x\to\pi/2^-}\left(\frac{a}{(\sin(x))^2+2}\right)=\lim_{x\to\pi/2^+}\left(\frac{bx+1}{x+1}\right)$$

$$\left(\frac{a}{1+2}\right)=\left(\frac{b\frac{\pi}{2}+1}{\frac{\pi}{2}+1}\right)$$

$$\frac{a}{3}=\frac{1+\frac{\pi}{2}b}{1+\frac{\pi}{2}}$$

   b. Find *a* and *b* so that the function is continuous and has the horizontal asymptote $y=2$ as $x\to\infty$.

$$\lim_{x\to\infty} f(x)=2$$

$$\lim_{x\to\infty}\left(\frac{bx+1}{x+1}\right)=2$$

$$\lim_{x\to\infty}\left(\frac{\frac{bx}{x}+\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}}\right)=2$$

$$\lim_{x\to\infty}\left(\frac{b+\frac{1}{x}}{1+\frac{1}{x}}\right)=2$$

$$b=2$$

$$\frac{a}{3}=\frac{1+\frac{\pi}{2}b}{1+\frac{\pi}{2}}$$

$$a=\frac{3\left(1+\frac{\pi}{2}(2)\right)}{1+\frac{\pi}{2}}$$

$$a=\frac{3+3\pi}{1+\frac{\pi}{2}}$$

Derivatives Solutions

1. Use the definition of the derivative (first principles) to calculate the derivative of the following:
   a. $f(x) = 5x + 4$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\big(5(x+h)+4\big)-(5x+4)}{h} \\ &= \lim_{h\to 0}\frac{5x+5h+4-5x-4}{h} \\ &= \lim_{h\to 0}\frac{5h}{h} \\ &= 5 \end{aligned}$$

   b. $f(x) = \dfrac{1}{2x+1}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\left(\frac{1}{2(x+h)+1}\right)-\left(\frac{1}{2x+1}\right)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{2x+2h+1}-\frac{1}{2x+1}}{h} \\ &= \lim_{h\to 0}\frac{\frac{(2x+1)-(2x+2h+1)}{(2x+1)(2x+2h+1)}}{h} \\ &= \lim_{h\to 0}\frac{\frac{2h}{(2x+1)(2x+2h+1)}}{h} \\ &= \lim_{h\to 0}\frac{2}{(2x+1)(2x+2h+1)} \\ &= \frac{2}{(2x+1)^2} \end{aligned}$$

c. $f(x)=\sqrt{x^2-3}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{(x+h)^2-3}\right)-\left(\sqrt{x^2-3}\right)}{h}\left(\frac{\sqrt{(x+h)^2-3}+\sqrt{x^2-3}}{\sqrt{(x+h)^2-3}+\sqrt{x^2-3}}\right)$$

$$=\lim_{h\to 0}\frac{((x+h)^2)-3)-(x^2-3)}{h\left(\sqrt{(x+h)^2-3}+\sqrt{x^2-3}\right)}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-3-x^2+3}{h\left(\sqrt{(x+h)^2-3}+\sqrt{x^2-3}\right)}$$

$$=\lim_{h\to 0}\frac{h(2x+h)}{h\left(\sqrt{(x+h)^2-3}+\sqrt{x^2-3}\right)}$$

$$=\lim_{h\to 0}\frac{(2x+h)}{\left(\sqrt{(x+h)^2-3}+\sqrt{x^2-3}\right)}$$

$$=\frac{2x}{2\sqrt{x^2-3}}$$

$$=\frac{2}{\sqrt{x^2-3}}$$

2. Find the derivatives of the following with respect to x:

a. $f(x)=\frac{1}{\sqrt{4}}e^{-(x-4)^2}$

$$f'(x)=\left(\frac{1}{\sqrt{4}}\right)\left(e^{-(x-4)^2}\right)\left(-2(x-4)\right)$$

$$=-e^{-(x-4)^2}(x-4)$$

b. $f(x)=\ln\left(\frac{1}{x^4+3}\right)$

$$f(x)=\ln(1)-\ln(x^4+2)$$

$$=-\ln(x^4+2)$$

$$f'(x)=-\left(\frac{1}{x^4+2}\right)\left(4x^3\right)$$

c. $f(x) = e^{2ax}\tan(bx) + \sin^2 x$

$$f(x) = e^{2ax}\tan(bx) + (\sin x)(\sin x)$$
$$f'(x) = \left(e^{2ax}\right)\left(2a\right)\left(\tan(bx)\right) + \left(e^{2ax}\right)\left(\sec^2(bx)\right)\left(b\right) + \left(\cos x\right)\left(\sin x\right) + \left(\sin x\right)\left(\cos x\right)$$
$$= \left(e^{2ax}\right)\left(2a\tan(bx) + b\sec^2(bx)\right) + 2\sin x\cos x$$

d. $f(x) = \ln\left(6e^{x^3}(x^3-3)^2\right)$ - two methods (chain rule or simplify starting function)

$$f(x) = \ln 6 + \ln e^{x^3} + \ln(x^3-3)^2$$
$$= \ln 6 + x^3 + 2\ln(x^3-3)$$
$$f'(x) = 3x^2 + 2\left(\frac{1}{x^3-3}\right)\left(3x^2\right)$$
$$= 3x^2 + \frac{6x^2}{x^3-3}$$

or

$$f'(x) = \frac{\left(6e^{x^3}(x^3-3)^2\right)'}{6e^{x^3}(x^3-3)^2}$$
$$= \frac{\left(6e^{x^3}\right)\left(3x^2\right)\left(x^3-3\right)^2 + \left(6e^{x^3}\right)2\left(x^3-3\right)\left(3x^2\right)}{6e^{x^3}(x^3-3)^2}$$

e. $f(x) = e^{(10x+11)}\ln(x^{2010}+1)$

$$f(x) = e^{(10x+11)}\ln(x^{2010}+1)$$
$$f'(x) = 10e^{(10x+11)}\ln(x^{2010}+1) + 2010e^{(10x+11)}\frac{1}{x^{2010}+1}$$
$$f'(x) = e^{(10x+11)}(10\ln(x^{2010}+1) + \frac{2010}{x^{2010}+1})$$

f. $f(x) = \cos^2(\sqrt{x})$

$$f(x) = \cos^2(\sqrt{x})$$
$$f'(x) = 2\cos(\sqrt{x})\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}$$

3. Consider the function $f(x) = \dfrac{1-x}{x-3}$

   a. Find the domain of f.

   $$D : \{x \in \mathbb{R}, x \neq 3\}$$

   b. Find the limits of f as x approaches $\pm\infty$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{1-x}{x-3}\right) = \lim_{x\to\pm\infty}\left(\frac{\frac{1}{x}-\frac{x}{x}}{\frac{x}{x}-\frac{3}{x}}\right) = \lim_{x\to\pm\infty}\left(\frac{\frac{1}{x}-1}{1-\frac{3}{x}}\right) = \frac{0-1}{1-0} = -1$$

c. Are there points where *f* is not continuous? If yes, find the left and right limit in each case.

f is not continuous at x = 3

$$\lim_{x\to 3^-}\frac{1-x}{x-3}\text{ , As } x\to 3^-, x-3<0 \text{ and } 1-x<0$$

$$\therefore \lim_{x\to 3^-}\frac{1-x}{x-3} = +\infty$$

$$\lim_{x\to 3^+}\frac{1-x}{x-3}\text{ , As } x\to 3^+, x-3>0 \text{ and } 1-x<0$$

$$\therefore \lim_{x\to 3^+}\frac{1-x}{x-3} = -\infty$$

d. Find the intervals where *f* is increasing and decreasing. Are there critical points?

$$f(x) = \frac{1-x}{x-3}$$

$$f'(x) = \frac{-(x-3)-(1-x)}{(x-3)^2}$$

$$= \frac{-x+3-1+x}{(x-3)^2}$$

$$= \frac{2}{(x-3)^2}$$

When x<3 or x>3, $(x-3)^2$ will always be positive.
Therefore *f* is increasing on $x \in (-\infty,2)\cup(2,+\infty)$
The critical point of *f* is x=2 since *f'(x)* is not defined at that point.

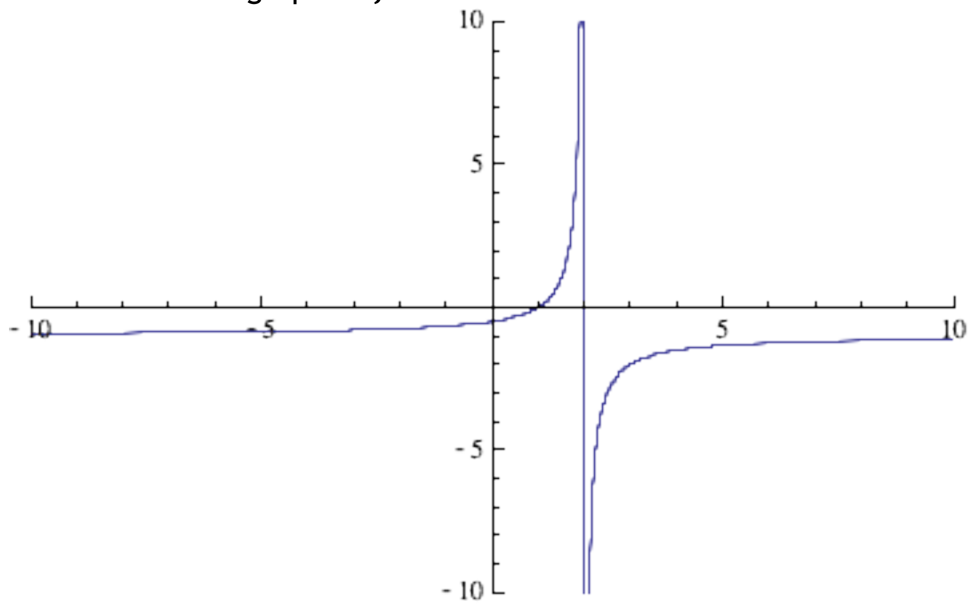
e. Find the intervals where *f* is concave up or concave down.

$$f'(x) = \frac{2}{(x-3)^2}$$

$$= 2(x-3)^{-2}$$

$$f''(x) = -4(x-3)^{-3}$$

f. Draw the graph of f.



7. Find the Taylor polynomial of degree 4 for $f(x)=\sin(2x)+x^2$ with base point $a = 0$. (x in radians!)

$$\begin{array}{ll} f(x)=\sin(2x)+x^2 & f(0)=0 \\ f^{I}(x)=2\cos(2x)+2x & f^{I}(0)=2 \\ f^{II}(x)=-4\sin(2x)+2 & f^{II}(0)=2 \\ f^{III}(x)=-8\cos(2x) & f^{III}(0)=-8 \\ f^{IV}(x)=16\sin(2x) & f^{IV}(0)=0 \end{array}$$

$$T_4(x)=\frac{f(0)(x-0)^0}{0!}+\frac{f^{I}(0)(x-0)^1}{1!}+\frac{f^{II}(0)(x-0)^2}{2!}+\frac{f^{III}(0)(x-0)^3}{3!}+\frac{f^{IV}(0)(x-0)^4}{4!}$$
$$=0+2x+\frac{2x^2}{2}-\frac{8x^3}{6}+0$$
$$=2x+x^2-\frac{4}{3}x^3$$

Integrals

1. Evaluate the following indefinite integrals:
   a. $\int(x^4+\cos x)\,dx$

$$\int(x^4+\cos x)\,dx$$
$$=\int x^4\,dx+\int\cos x\,dx$$
$$=\frac{1}{5}x^5+\sin x+C$$

b. $\int \frac{e^{3x}+4}{e^{3x}}dx$

$$\int \frac{e^{3x}+4}{e^{3x}}dx$$
$$= \int \frac{e^{3x}}{e^{3x}} + \frac{4}{e^{3x}}dx$$
$$= \int 1dx + \int \frac{4}{e^{3x}}dx$$
$$= x - \frac{4}{3}e^{-3x} + C$$

c. $\int 16x^3 \ln(8x)dx$ - Integration by parts

$$u = \ln(8x) \quad du = \frac{1}{x}$$
$$dv = 16x^3 \quad v = 4x^4$$

$$\int 16x^3 \ln(8x)dx$$
$$= 4x^4 \ln(8x) - \int 4x^3 dx$$
$$= 4x^4 \ln(8x) - 4\left(\frac{x^4}{4}\right) + C$$
$$= x^4(4\ln(8x) - 1) + C$$

d. $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx$ - Integration by subsitution

$$u = \frac{1}{x}$$
$$du = -\frac{1}{x^2}dx$$

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx$$
$$= \int \frac{\sin u}{x^2}(-x^2)du$$
$$= -\int \sin u \, du$$
$$= -(-\cos u) + C$$
$$= \cos\left(\frac{1}{x}\right) + C$$

e. $\int \ln x\, dx$ - Integration by parts

$$\begin{aligned}&\int \ln x\, dx\\&= \int 1\cdot \ln x\, dx\\&= x\ln x - \int \frac{1}{x} x\, dx\\&= x\ln x - x + C\end{aligned}$$

$u = \ln x \quad du = \frac{1}{x}$

$dv = 1 \quad v = x$

f. $\int e^x \cos x\, dx$ - Integration by parts x 2

$$\int e^x \cos x\, dx = e^x \cos x - \int -e^x \sin x\, dx$$

$u = \cos x \quad du = -\sin x$

$dv = e^x \quad v = e^x$

$$\int e^x \cos x\, dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$$

$u = \sin x \quad du = \cos x$

$dv = e^x \quad v = e^x$

$$2\int e^x \cos x\, dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x\, dx = \frac{1}{2}(e^x \cos x + e^x \sin x)$$

2. An athlete starts a marathon with a speed of 14 km/h. Due to an existing ankle injury she is forced to slow down and her speed decreases at a constant rate according to the differential equation $v'(t) = -2, t \geq 0$ with $v(0) = 15$. Answer the following questions:
   a. Find the equation for the speed in km/h as a function of time in hours.

$$\begin{aligned}v'(t) &= -2\\ \int v'(t)\, dt &= \int -2\, dt\\ v(t) &= -2t + C\\ v(0) &= -2(0) + C\\ 15 &= C\end{aligned}$$

$$\therefore v(t) = -2t + 15$$

   b. Solve the pure-time differential equation $\frac{dp}{dt} = v(t), p(0) = 0$ for the location p in km.

$$\begin{aligned}&\frac{dp}{dt} = v(t)\\ &\int \frac{dp}{dt} dt = \int (-2t + 15) dt\\ &p(t) = -t^2 + 15t + A\\ &p(0) = -(0)^2 + 15(0) + A\\ &0 = A\end{aligned}$$

$$\therefore p(t) = -t^2 + 15t$$

   c. How long will it take the athlete to complete the marathon (42 km)?

$$p(t) = -t^2 + 15t$$
$$42 = -t^2 + 15t$$
$$0 = -t^2 + 15t - 42$$

$$t = \frac{-15 \pm \sqrt{15^2 - 4(-1)(-42)}}{2(-1)}$$
$$= \frac{-15 \pm \sqrt{57}}{-2}$$

$$\therefore t_1 = 3.725, t_2 = 11.275$$

It will take the athlete 3.725 hours to complete the marathon.