**University of Ottawa**
**ECO 3145 Mathematical Economics I**
**Fall 2014, Professor Shiell**

**Exam I – Answers**

1. $L + 2W = 32 \quad \Rightarrow \quad L = 32 - 2W$

and

$A = LW$

Substitution:

$$A = (32 - 2W)W$$
$$= 32W - 2W^2$$

Choose W to maximize A. This requires finding the stationary point.

$$\frac{\partial A}{\partial W} = 32 - 4W = 0 \quad \Rightarrow \quad W = 8$$

Substitution:

$L = 32 - 2W = 32 - 2(8) = 16$.

So the stationary point is $(W = 8, L = 16)$.

To check whether it is a local maximum or minimum, check the second derivative at the stationary point.

$$\frac{\partial^2 A}{\partial W^2} = -4 < 0$$

Since the second derivative is negative, the stationary point is a local maximum.

(Actually, since the derivative has the same sign regardless of the point at which it is evaluated, we can conclude that the stationary point is a global maximum.)

2. Consider the function $f(x_1, x_2) = 100 - 5x_1 + 4x_1^2 - 9x_2 + 5x_2^2 + 8x_1x_2$.

a.) Derive the first-order conditions.

$$f_1 = -5 + 8x_1 + 8x_2 = 0$$

$$f_2 = -9 + 10x_2 + 8x_1 = 0$$

Rewriting:

$$8x_1 + 8x_2 = 5$$
$$8x_1 + 10x_2 = 9$$

In matrix form:

$$\begin{bmatrix} 8 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Call the coefficient matrix A, i.e. $A = \begin{bmatrix} 8 & 8 \\ 8 & 10 \end{bmatrix}$.

Now solve for $x_1$ and $x_2$ using Cramer's rule (alternatively you can compute the inverse matrix $A^{-1}$ and solve the system for the x vector).

Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} 5 & 8 \\ 9 & 10 \end{vmatrix}}{|A|} \qquad x_2 = \frac{\begin{vmatrix} 8 & 5 \\ 8 & 9 \end{vmatrix}}{|A|}$$

Note $|A| = 80 - 64 = 16$

Therefore:

$$x_1 = \frac{50 - 72}{16} = -\frac{11}{8} \qquad x_2 = \frac{72 - 40}{16} = \frac{32}{16} = 2$$

b.) Construct the Hessian matrix:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

$f_{11} = 8$ $\quad$ $f_{12} = 8$ $\quad$ $f_{21} = 8$ $\quad$ $f_{22} = 10$

$$H = \begin{bmatrix} 8 & 8 \\ 8 & 10 \end{bmatrix}$$

$LPM_1 = |8| = 8$

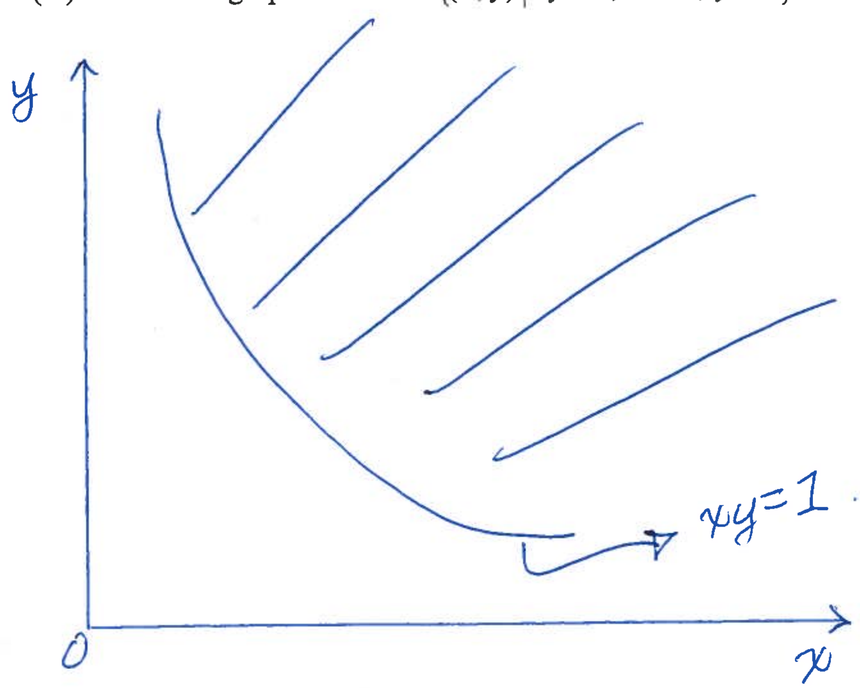
$LPM_2 = |H| = 80 - 64 = 16$

Since both LPM's are positive, the H matrix is positive definite.

Therefore the stationary point represents a local minimum.

(Actually, since the LPM's have the same signs regardless of the point at which they are evaluated, the function must be globally convex. Therefore the stationary point represents a global minimum.)

3. (X) Sketch the graph of the set $\{(x, y) \mid xy \geq 1,\ x > 0,\ y > 0\}$. Is this set convex? Why?




Yes, this set is convex. Consider any two points, a and b, within this set and the convex combination. $\lambda a + (1-\lambda) b$, for $\lambda \in [0,1]$ The convex combination is everywhere inside the set. Therefore, the set is convex.

4. a.) Revising the notation: $f(x_1, x_2) = 2x_1^2 - x_1x_2 + x_2^2$

Consider any two points: $\overline{x} = (\overline{x}_1, \overline{x}_2)$ and $\hat{x} = (\hat{x}_1, \hat{x}_2)$.

The function is concave if $\lambda f(\overline{x}) + (1-\lambda) f(\hat{x}) \le f(\lambda\overline{x} + (1-\lambda)\hat{x})$

$$f(\overline{x}) = 2\overline{x}_1^2 - \overline{x}_1\overline{x}_2 + \overline{x}_2^2$$

$$f(\hat{x}) = 2\hat{x}_1^2 - \hat{x}_1\hat{x}_2 + \hat{x}_2^2$$

$$\lambda\overline{x} + (1-\lambda)\hat{x} = (\lambda\overline{x}_1 + (1-\lambda)\hat{x}_1, \lambda\overline{x}_2 + (1-\lambda)\hat{x}_2)$$

Then the right-hand side (RHS) of the formula becomes:

$$f(\lambda\overline{x} + (1-\lambda)\hat{x}) = 2[\lambda\overline{x}_1 + (1-\lambda)\hat{x}_1]^2 - (\lambda\overline{x}_1 + (1-\lambda)\hat{x}_1)(\lambda\overline{x}_2 + (1-\lambda)\hat{x}_2) + [\lambda\overline{x}_2 + (1-\lambda)\hat{x}_2]^2$$

And the left-hand side (LHS) of the formula becomes:

$$\lambda(2\overline{x}_1^2 - \overline{x}_1\overline{x}_2 + \overline{x}_2^2) + (1-\lambda)(2\hat{x}_1^2 - \hat{x}_1\hat{x}_2 + \hat{x}_2^2).$$

So we have the indeterminate relation LHS ? RHS.

Now we must manipulate the LHS and RHS in order to determine the nature of the relation.

If LHS $\le$ RHS, then the function is concave.

If LHS $<$ RHS, then the function is strictly concave.

If LHS $\ge$ RHS, then the function is convex.

If LHS $>$ RHS, then the function is weakly convex.

b.) If the function is concave, then the stationary point is a global maximum.

If the function is strictly concave, then the stationary point is a unique global maximum.

If the function is convex, then the stationary point is a global minimum.

If the function is strictly convex, then the stationary point is a unique global minimum.

5. Consider the equation $x^2 + 3xy + 2yz + y^2 + z^2 - 11 = 0$.

a.) First note that the equation has the form $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + 3xy + 2yz + y^2 + z^2 - 11$.

Now check that the three conditions of the Implicit Function Theorem are satisfied.

First, does F have continuous partial derivatives?

$F_x = 2x + 3y$ $\qquad$ $F_y = 3x + 2z + 2y$ $\qquad$ $F_z = 2y + 2z$

All three derivatives are continuous.

Second, is the equation satisfied at the point? Yes.

Third, is $F_z \neq 0$ at the point? $F_z(1,2,0) = 2(2) + 2(0) = 4$

Since all three conditions are satisfied, it follows that $z = f(x, y)$ is implicitly defined at the given point.

b.) $$\frac{dz}{dx} = -\frac{F_x(1,2,0)}{F_z(1,2,0)} = -\frac{2(1) + 3(2)}{4} = -2$$