Concordia University
Engineering and
Computer Science

FACULTY OF ENGINEERING AND COMPUTER SCIENCE

COURSE		NUNMBER	SECTION
Numerical Methods in Engineering		**ENGR 391**	**/2 V**
EXAMINATION **Midterm Exam**	DATE **March 1st, 2012**	TIME **13:15 – 14:30**	# of pages (including the title page) **8**
PROFESSOR **Dr. Liangzhu (Leon) Wang**			

MATERIALS ALLOWED – None (closed book, closed notes)

CALCULATORS ALLOWED – Yes (Faculty approved calculators)

SPECIAL INSTRUCTIONS:

Write your name and ID clearly;
Read each question carefully;
Answer all four questions;
Write down clearly all your intermediate steps;
Write on the back of each page for extra space.

Name: X X X X X X **ID:** X X X X X X

Signature: X X X X X X

Question #1 [10 pts]

Given the following function: $f(x) = x \cdot (\sqrt{x+1} - \sqrt{x})$

a) Use four-digit rounding arithmetic to evaluate *f(500)*. (3 pts)

b) When $x = 500$, which is a large number, *f(500)* is going to have large round-off errors due to subtractive cancellation.

   To reduce the round-off errors, try to reformulate *f(x)* and use four-digit rounding arithmetic to evaluate *f(500)*. (4 pts)

c) Calculate the percentage relative errors in part a) and b) by using the true value of *f(500) = 11.1747553*. (3 pts).

a) $f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)$

$= 500 \cdot \left(\sqrt{501} - \sqrt{500}\right)$

$= 500 \cdot (22.38 - 22.36)$

$= 500 \cdot (0.02000) = 10.00$

(3)

b) $f(x) = x \cdot \left(\sqrt{x+1} - \sqrt{x}\right) \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$

$= \frac{x}{\sqrt{x+1} + \sqrt{x}} = \frac{500}{\sqrt{501} + \sqrt{500}} = \frac{500}{22.38 + 22.36}$

$= \frac{500}{44.74} = 11.18$

(4)

c) Percent relative error =

a) $\left|\frac{11.1747553 - 10.00}{11.1747553}\right| \times 100\% = 10.51\%$

b) $\left|\frac{11.1747553 - 11.18}{11.1747553}\right| \times 100\% = 0.046\ \%$

(3)

Question #2 [10 pts]

a) Find the Taylor series expansion of $\cos(x)$ when $x_i = 0$ **(keep up to x^{12})**. (3 pts)

Taylor series:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f'''(x_i)}{3!}(x_{i+1} - x_i)^3 + \ldots + \frac{f^n(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

b) Use the first three terms from part a) to calculate the value of $\cos(\frac{\pi}{3})$;
And calculate the true percent relative error.
(Keep two digits after the decimal point and apply rounding for the percent number). (4 pts)

c) Prove that if we use the first ***2k+1 (k=1, 2, 3…)*** terms to calculate the value of $\cos(\frac{\pi}{3})$, we always overestimate the value of $\cos(\frac{\pi}{3})$ (3 pts)

For example, we use the first three, or five, or seven, or nine …. terms to calculate $\cos(\frac{\pi}{3})$.

a) $\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$ (3)

b) $\cos(\frac{\pi}{3}) \approx 1 - (\frac{1.04720}{2!})^2 + \frac{(1.04720)^4}{4!}$

$= 1 - \frac{1.09663}{2} + \frac{1.20259}{24}$

$= 1 - 0.548315 + 0.0501079$

$= 0.501793$ (2)

Exact: $\cos(\frac{\pi}{3}) = \frac{1}{2}$

$\varepsilon = \left| \frac{0.5 - 0.501793}{0.5} \right| \approx 0.36\%$ (2)

c) $\cos(x) \approx 1 - (\frac{x^2}{2!} - \frac{x^4}{4!}) - (\frac{x^6}{6!} - \frac{x^8}{8!}) - (\frac{x^{10}}{10!} - \frac{x^{12}}{12!})$ (1)

the terms in the brackets:

$\frac{x^{2k}}{(2k)!} - \frac{x^{2k+2}}{(2k+2)!}$ $(k = 1, 3, 5 \cdots)$

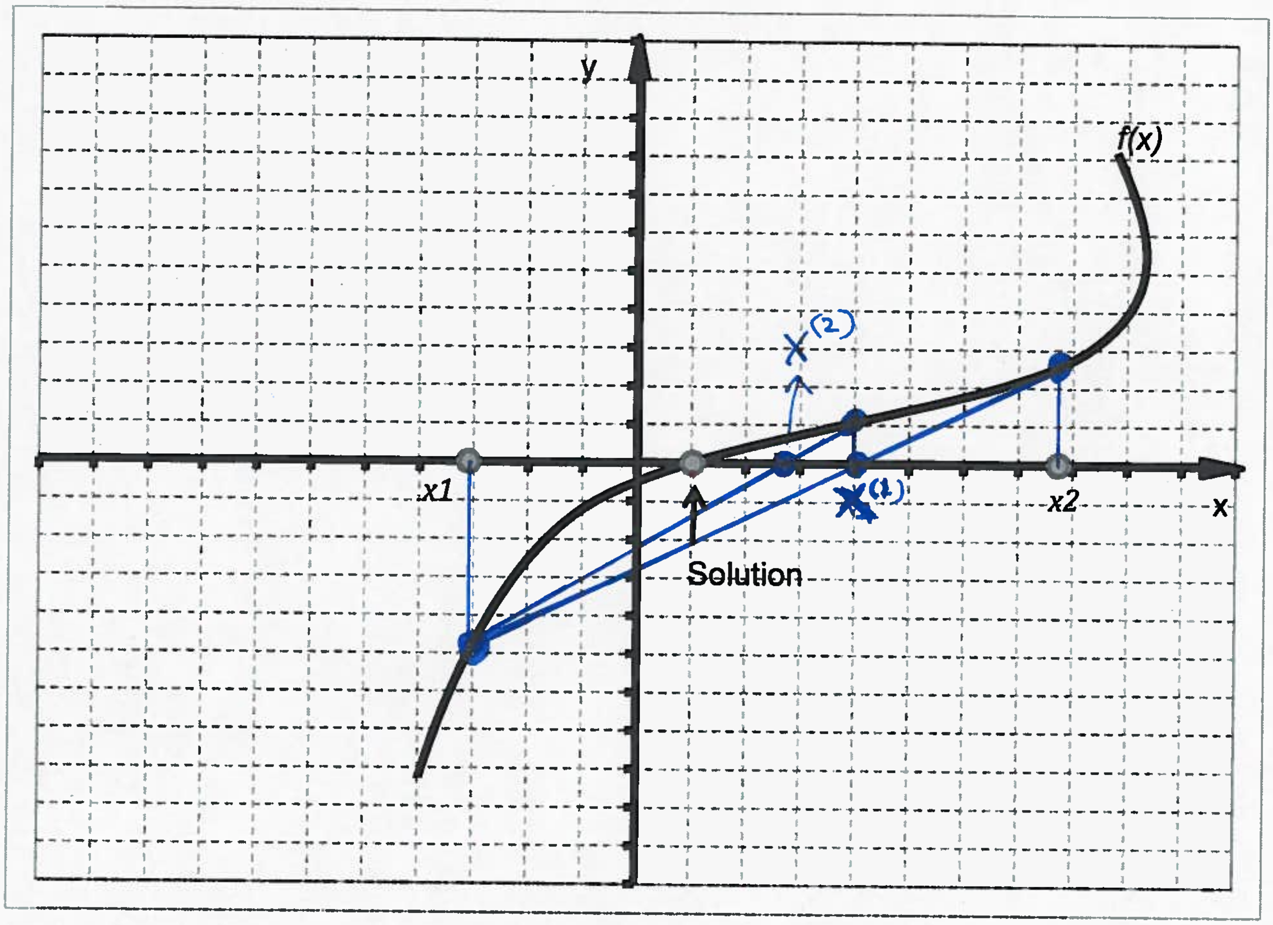
when $x = \frac{\pi}{3} \approx 1.04720$: $\frac{x^{2k}}{(2k)!} > \frac{x^{2k+2}}{(2k+2)!}$

so: $\frac{x^2}{2!} - \frac{x^4}{4!} > 0$; $\frac{x^6}{6!} - \frac{x^8}{8!} > 0$; ….. (2)

The first $2k+1$ $(k=1, 2, 3 \cdots)$ always overestimate $\cos(\frac{\pi}{3})$

Question #3 [15 pts]

a) Given two initial guesses $x1$ and $x2$, illustrate graphically the False Position Method to solve $f(x) = 0$ in the following graph (for two iterations) (3 pts)

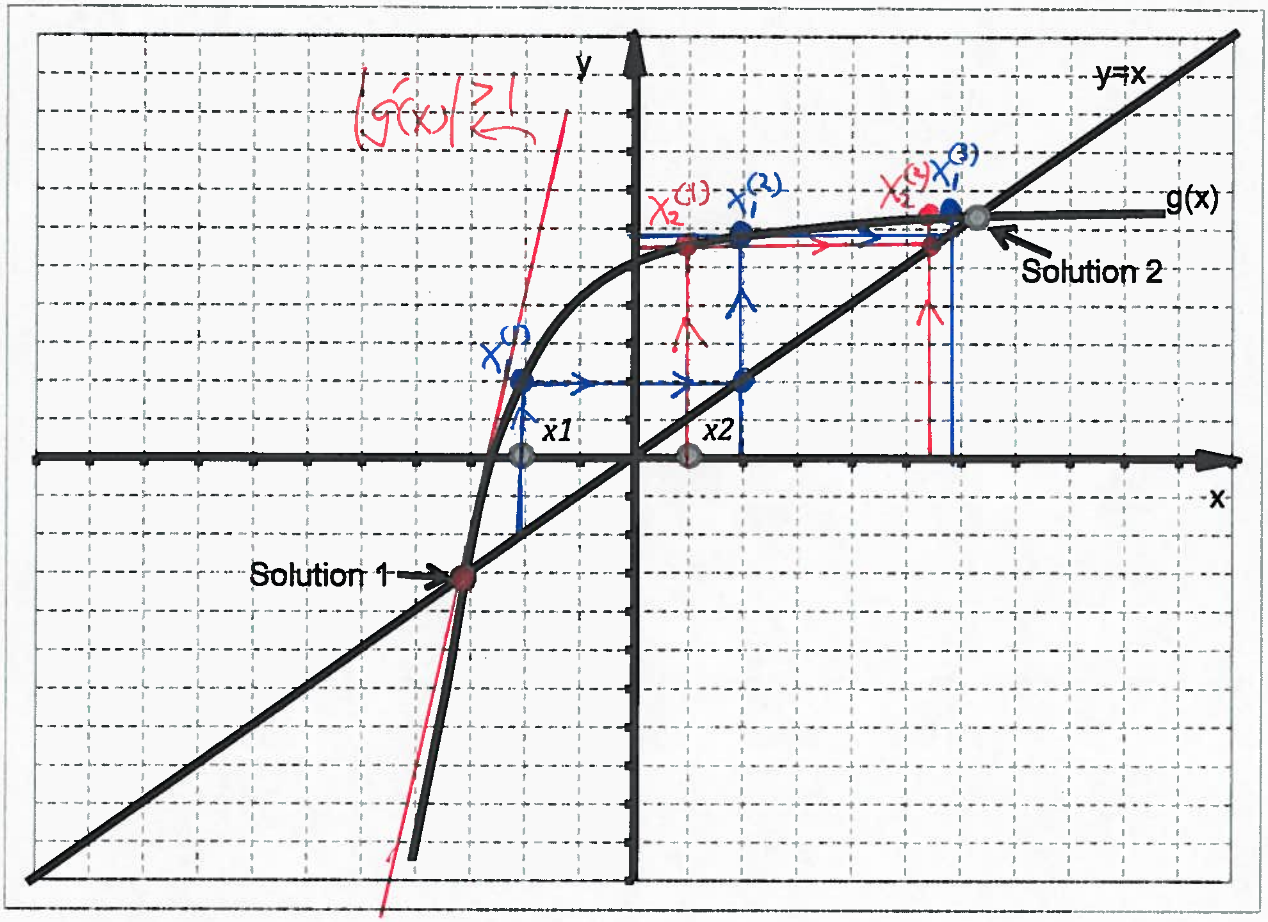


b) An ENGR 391 student is trying to solve $f(x) = 0$ by using the Fixed-point Iteration Method. He finds the function, $g(x)$ and plots it in the following graph. Apparently, there are two solutions.

i) Will he find **Solution 1** by using the point $x1$? Illustrate your explanations on the graph with two iterations. (3 pts)

ii) Will he find **Solution 2** by using the point $x2$? Illustrate your explanations on the graph with two iterations. (3 pts)

iii) By using the $g(x)$, that he currently selects, can he find **Solution 1** by any means? If not, why? (3 pts)



i) $x_{i+1} = g(x_i)$; NO, divergent solution

ii) $x_{i+1} = g(x_i)$; Yes, convergent solution

iii) NO, because near solution 1, $|g'(x)| > 1$
he will never find a solution.

c) If we want to solve $\cos(x) = \sin(x)$ by using the Fixed-point Iteration Method for $x \in [0,1]$

Find an appropriate function, ***g(x)***, to give a convergent solution of ***x*** and justify your answer (you do not need to solve it). (3 pts)

$$x + \cos(x) = x + \sin(x)$$

$$x = x + \cos(x) - \sin(x)$$

$$g(x) = x + \cos(x) - \sin(x) \qquad (1)$$

$$g'(x) = 1 - \sin(x) - \cos(x)$$

$$|g'(0)| = |1 - 0 - 1| = 0 < 1 \qquad (1)$$

$$|g'(1)| = |1 - \sin(1) - \cos(1)| \qquad (1)$$

$$= |1 - 0.8415 - 0.5403|$$

$$= |-0.3818| < 1$$

so: g(x) gives convergent solution.

Question #4 [15 pts]

Given following matrix equation

$$\begin{cases} 2x_1 + x_2 - x_3 = 0 \\ 5x_1 + 2x_2 + 2x_3 = 3 \\ 3x_1 + 2x_2 + x_3 = 1 \end{cases}$$

a) Find the **LU decomposition** of the coefficient matrix **[A] with pivoting (i.e. PA=LU)**. (6 pts)

b) Find the solution of the above matrix equation using the **LU decomposition with pivoting** (for decimal numbers, keep **4** significant digits for the final result). (4 pts)

c) Using the **LU decomposition** obtained in part a), find the **second** column of $[A]^{-1}$ (5 pts)

$$\begin{bmatrix} 2 & 1 & -1 \\ 5 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot: $\rightarrow \begin{bmatrix} 5 & 2 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) $\rightarrow \begin{bmatrix} 5 & 2 & 2 \\ (2/5) & 0.2 & -1.8 \\ (3/5) & 0.8 & -0.2 \end{bmatrix} \qquad \begin{bmatrix} 5 & 2 & 2 \\ 0.4 & 0.2 & -1.8 \\ 0.6 & 0.8 & -0.2 \end{bmatrix}$

6

pivot $\rightarrow \begin{bmatrix} 5 & 2 & 2 \\ (3/5) & 0.8 & -0.2 \\ (2/5) & 0.2 & -1.8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 5 & 2 & 2 \\ (3/5) & 0.8 & -0.2 \\ (2/5) & (1/4) & -1.75 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 5 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (3/5) & 1 & 0 \\ (2/5) & (1/4) & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 & 2 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & -1.75 \end{bmatrix}$ $\quad \begin{bmatrix} 5 & 2 & 2 \\ 0 & \frac{4}{5} & -\frac{1}{5} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.4 & 0.25 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 2 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & -1.75 \end{bmatrix}$$

(2) (2)

$LZ = Pb$

(2) $$\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.4 & 0.25 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -0.8 \\ -1 \end{bmatrix}$$

$UX = Z$

(2) $$\begin{bmatrix} 5 & 2 & 2 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & -1.75 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -0.8 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7143 \\ -0.8571 \\ 0.5714 \end{bmatrix} \begin{matrix} 5/7 \\ -6/7 \\ 4/7 \end{matrix}$$

Second column of $[A]^{-1}$

$$LZ = P \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

(5) $$\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.4 & 0.25 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.6 \\ -0.25 \end{bmatrix} \quad (2)$$

$UX = Z$

$$\begin{bmatrix} 5 & 2 & 2 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & -1.75 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.6 \\ -0.25 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.4286 \\ -0.7142 \\ 0.1429 \end{bmatrix} \begin{matrix} 3/7 \\ -5/7 \\ 1/7 \end{matrix} \quad (2)$$

Total Grade: 50

Your Grade: XXXXXX