CONCORDIA UNIVERSITY
Department of Mechanical and Industrial Engineering
ENGR 311/2 X; Transform Calculus and Partial Differential Equations
Instructor: Dr C. Rajalingham
Midterm Test

Friday, October 25, 2013 **08:45-10:00**

Instructions:

1. Do not unstaple the question paper
2. Answer all questions. Show all important steps of your answer. Credit will be given to neat and unambiguous answers.
3. Only the Faculty approved Sharp 531 or Casio FX-300 MS calculators are allowed. Cell phones and other electronic devices are prohibited.

Question 1:

(a) Determine the Laplace transform of the following functions. You need not have to simplify the results.

(i) $f(t) = t^2 \cosh 3t$

Hint: $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$

(ii) $f(t) = (3\cos 2t + 4\cos 3t)\, u\left(t - \frac{\pi}{2}\right)$

(b) Laplace transform the differential equation $y'' + 4y' + 5y = \delta(t-1) + 5u(t-1)$ subjected to the initial conditions $y(0) = -1$ and $y'(0) = 1$. Provide the expression for $Y(s)$. You need not have to solve this transformed equation.

Question 2:

(a) Determine the inverse Laplace transforms of the following functions

(i) $F(s) = \dfrac{4(s+1)}{s^2\left(s^2+4\right)}$

(ii) $F(s) = \dfrac{4(s+5)}{\left(s^2+2s+5\right)^2}$

(b) :Laplace transform of a function $f(t)$, which is defined for $t \geq 0$, is given as

$$F(s) = \frac{1}{s^2} - \frac{2\,e^{-s}}{s} - \frac{e^{-2s}}{s^2}\ .$$

(i) Express $f(t)$ in terms of delayed step functions.

(ii) Sketch the graph of $f(t)$

Question 3:

Periodic function is defined in one period as

$$f(x)=\begin{cases}0 & -2\prec x\prec 0\\ x^2 & 0\prec x\prec 1\\ 1 & 1\prec x\prec 2\end{cases}$$

(a) Provide a sketch of its graph in the range $-6\prec x\prec 6$. Define the value of this function at $x=-2,\,0,\,1\;and\;2$.

(b) Express the Fourier coefficients as integrals..

(c) Determine the Fourier coefficients

Question 4:.

An odd-periodic function $f(x)$ is defined in the half-period $0\prec x\prec 1$ as $f(x)=2x-x^2$.

(a) Provide the expression for the function in the remaining half period $-1\prec x\prec 0$. Define its value at $x=-1,\,0\;and\;1$.

(b) Sketch the graph of this function in the full period $-1\prec x\prec 1$.

(c) Express the Fourier coefficients as integrals

(d) Determine the Fourier coefficients.

. Laplace Transform Table:

f(t)	F(s)
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
e^{at}	$\frac{1}{(s-a)}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}\, t^n$	$\frac{n!}{(s-a)^{n+1}}$
$\sinh \omega t$	$\frac{\omega}{(s^2-\omega^2)}$
$\cosh \omega t$	$\frac{s}{(s^2-\omega^2)}$
$\sin \omega t$	$\frac{\omega}{(s^2+\omega^2)}$
$\cos \omega t$	$\frac{s}{(s^2+\omega^2)}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at} \cos \omega t$	$\frac{(s-a)}{(s-a)^2+\omega^2}$
$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
$\omega t \sin \omega t$	$\frac{2\omega^2 s}{(s^2+\omega^2)^2}$

Laplace Transform Properties:

$f(t)$	$F(s)$
$\alpha\, f(t)+\beta\, g(t)$	$\alpha\, F(s)+\beta\, G(s)$
$e^{at} f(t)$	$F(s-a)$
$f'(t)$	$s\,F(s)-f(0)$
$f''(t)$	$s^2\, F(s)-s\, f(0)-f'(0)$
$f^{(n)}(t)$	$s^n\, F(s)-s^{n-1}\, f(0)-s^{n-2} f'(0)- \quad -f^{(n-1)}(0)$
$\int_0^t f(\tau)d\tau$	$\dfrac{F(s)}{s}$
$-t\, f(t)$	$F'(s)$
$f(t-a)u(t-a)$	$e^{-as}\, F(s)$
$\int_0^t f(\tau)\, g(t-\tau)d\tau$	$F(s)G(s)$

Fourier Series:

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=0}^{\infty}\left\{a_n \cos\frac{n\pi x}{p}+b_n \sin\frac{n\pi x}{p}\right\}$$

$$a_0=\frac{1}{p}\int_{-p}^{p} f(x)\, dx$$

$$a_n=\frac{1}{p}\int_{-p}^{p} f(x)\, \cos\frac{n\pi x}{p}\, dx$$

$$b_n=\frac{1}{p}\int_{-p}^{p} f(x)\, \sin\frac{n\pi x}{p}\, dx$$

ENGR 311/2 X : Transform Calculus and Partial Differential Equations.

1 (a) (i) $f(t) = t^2 \cosh 3t$

$$= t^2 \left(\frac{e^{3t} + e^{-3t}}{2} \right). \qquad \mathcal{L}\{t^2\} = \frac{2}{s^3}.$$

$$\therefore F(s) = \mathcal{L}\left\{\tfrac{1}{2}e^{3t}t^2 + \tfrac{1}{2}e^{-3t}t^2\right\}$$

$$= \frac{1}{2}\cdot\frac{2}{(s-3)^3} + \frac{1}{2}\cdot\frac{2}{(s+3)^3}.$$

$$= \frac{1}{(s-3)^3} + \frac{1}{(s+3)^3}$$

(ii) $f(t) = g(t)\cdot u(t-\frac{\pi}{2}) \qquad g(t) = 3\cos 2t + 4\cos 3t.$

$$= h(t-\tfrac{\pi}{2})\, u(t-\tfrac{\pi}{2})$$

Now: $h(t-\frac{\pi}{2}) = g(t)$

$$\therefore h(t) = g(t+\tfrac{\pi}{2})$$

$$= 3\cos 2(t+\tfrac{\pi}{2}) + 4\cos 3(t+\tfrac{\pi}{2})$$

$$= 3\cos(\pi + 2t) + 4\cos(\tfrac{3\pi}{2} + 3t)$$

$$= -3\cos 2t + 4\sin 3t.$$

$$\therefore H(s) = -3.\frac{s}{s^2+4} + 4\frac{3}{s^2+3^2}$$

$$= \left(-\frac{3s}{s^2+4} + \frac{12}{s^2+9}\right).$$

$$\therefore F(s) = e^{-\frac{\pi}{2}s}. H(s)$$

$$= e^{-\frac{\pi}{2}s}.\left(-\frac{3s}{s^2+4} + \frac{12}{s^2+9}\right)$$

(b) $y'' + 4y' + 5y = \delta(t-1) + 5u(t-1)$

$y(0) = -1, \quad y'(0) = 1.$

Transforming:

$$\{s^2Y - s(-1) - (1)\} + 4\{sY - (-1)\} + 5Y = e^{-s} + \frac{5}{s}e^{-s}$$

$$(s^2+4s+5)Y + (s+3) = e^{-s}\left(1+\frac{5}{s}\right)$$

$$\therefore Y = -\frac{(s+3)}{(s^2+4s+5)} + e^{-s}\frac{(s+5)}{s(s^2+4s+5)}$$

2 (a) (i) $F(s) = \dfrac{4(s+1)}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}$

$B = \dfrac{4(1)}{(4)} = 1.$

$4s + 4 = As(s^2+4) + B(s^2+4) + s^2(Cs+D)$

Coef s^2 $\quad 0 = B + D \qquad D = -1.$

Coef s $\quad 4 = 4A \qquad A = 1$

Coef s^3 $\quad 0 = A + C \qquad C = -1.$

$\therefore F(s) = \dfrac{1}{s} + \dfrac{1}{s^2} - \dfrac{s+1}{s^2+4}$

$\therefore f(t) = 1 + t - \cos 2t - \frac{1}{2}\sin 2t.$

(ii) $F(s) = \dfrac{4(s+5)}{(s^2+2s+5)^2}$

$= \dfrac{4(s+5)}{((s+1)^2+4)^2} = \dfrac{4(s+1)+16}{((s+1)^2+2^2)^2}$

$\therefore F(s) = G(s+1) \qquad G(s) = \dfrac{4s+16}{(s^2+2^2)^2}$

$f(t) = e^{-t} g(t)$

$= \dfrac{1}{2} \cdot \dfrac{2 \cdot 2^2 s}{(s^2+2^2)^2} + \dfrac{2 \cdot 2^3}{(s^2+2^2)^2}$

$\therefore g(t) = \frac{1}{2}(2t \sin 2t)$
$\quad + (\sin 2t - 2t\cos 2t)$

$\therefore f(t) = e^{-t}\{t \sin 2t + \sin 2t - 2t \cos 2t\}$

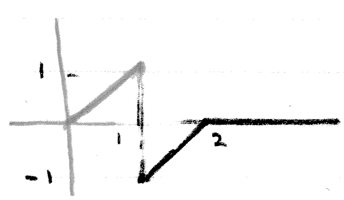
(b) $F(s) = \dfrac{1}{s^2} - \dfrac{2}{s}e^{-s} - \dfrac{1}{s^2}e^{-2s}$

(i) $\therefore f(t) = t - 2u(t-1) - (t-2)u(t-2)$

(ii) $0 < t < 1 \qquad f(t) = t$

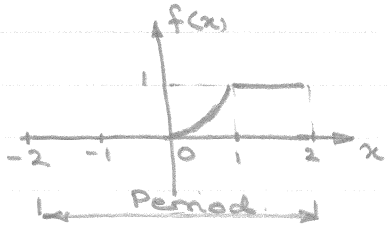
$1 < t < 2 \qquad f(t) = t - 2$

$2 < t \qquad f(t) = t - 2 - (t-2) = 0$



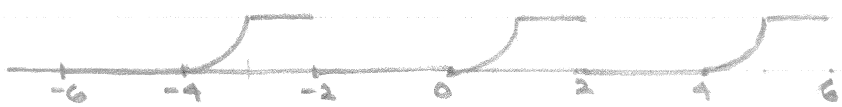
3.

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 < x < 2 . \end{cases}$$



$2p = 4 \qquad p = 2$

(a)



3 periods

$$f(-2) = \tfrac{1}{2}(1+0) = \tfrac{1}{2}$$
$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = \tfrac{1}{2}(1+0) = \tfrac{1}{2}$$

(b)

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$
$$= \frac{1}{2}\left\{ \cancel{\int_{-2}^{0} 0\,dx} + \int_0^1 x^2\,dx + \int_1^2 1\,dx \right\}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{2}\left\{ \cancel{\int_{-2}^{0} 0\,dx} + \int_0^1 x^2 \cos\frac{n\pi x}{2}\,dx + \int_1^2 \cos\frac{n\pi x}{2}\,dx \right\}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{2}\left\{ \cancel{\int_{-2}^{0} 0\,dx} + \int_0^1 x^2 \sin\frac{n\pi x}{2}\,dx + \int_1^2 \sin\frac{n\pi x}{2}\,dx \right\}$$

(c)

$$a_0 = \frac{1}{2}\left\{ 0 + \left[\frac{x^3}{3}\right]_0^1 + \left[x\right]_1^2 \right\}$$
$$= \frac{1}{2}\left\{ \frac{1}{3} - 0 + 2 - 1 \right\} = \frac{2}{3}$$

$$a_n = \frac{1}{2}\left\{ \int_0^1 x^2 \cos\frac{n\pi x}{2}\,dx + \int_1^2 \cos\frac{n\pi x}{2}\,dx \right\}$$
$$= \frac{1}{2}\left\{ \left[x^2 \cdot \frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^1 - \int_0^1 2x \cdot \frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx + \left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_1^2 \right\}$$
$$= \frac{1}{2}\left\{ \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} - \frac{4}{n\pi}\int_0^1 x\sin\frac{n\pi x}{2}\,dx + \cancel{\frac{2}{n\pi}\sin n\pi} - \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} \right\}$$
$$= -\frac{2}{n\pi}\left\{ \left[-x \cdot \frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^1 - \int_0^1 -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx \right\}$$
$$= \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left\{ \frac{2}{n\pi}\sin\frac{n\pi x}{2} \right\}_0^1$$
$$= \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^3\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\left\{\int_0^1 x^2 \sin\frac{n\pi x}{2}\,dx + \int_1^2 \sin\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{\left[-x^2\cdot\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^1 - \int_0^1 -2x\cdot\frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx + \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2\right\}$$

$$= \left\{-\frac{1}{n\pi}\cancel{\cos\frac{n\pi}{2}} + \frac{2}{n\pi}\int_0^1 x\cos\frac{n\pi x}{2}\,dx - \frac{1}{n\pi}\cos n\pi + \frac{1}{n\pi}\cancel{\cos\frac{n\pi}{2}}\right\}$$

$$= -\frac{1}{n\pi}\cos n\pi + \frac{2}{n\pi}\left\{\left[x\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^1 - \int_0^1 1\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right\}$$

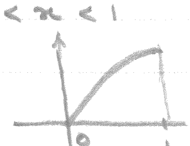
$$= -\frac{1}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^1$$

$$= -\frac{1}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{8}{n^3\pi^3}\left(\cos\frac{n\pi}{2} - 1\right)$$

4) Odd-function extension:

Half period: $0 < x < 1$ $\quad f(x) = g(x)$ $\quad g(x) = 2x - x^2$

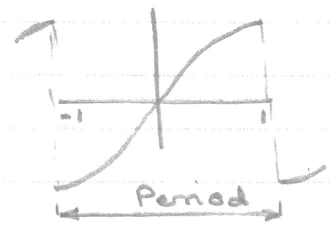
$g'(x) = 2 - 2x$



(a) Other Half period: $-1 < x < 0$ $\quad f(x) = -g(-x)$

$= -\{2(-x) - (-x)^2\}$

$= 2x + x^2.$



$p = 1.$

(b) $f(-1) = \frac{1}{2}(1-1) = 0$

$f(0) = 0$

$f(1) = \frac{1}{2}(1-1) = 0$

(c) For odd function $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$

$$b_n = \frac{2}{1}\cdot\int_0^1 (2x - x^2)\sin n\pi x\,dx.$$

(d)
$$b_n = 2\left\{\left[-(2x-x^2)\cdot\frac{1}{n\pi}\cos n\pi x\right]_0^1 - \int_0^1 -(2-2x)\cdot\frac{1}{n\pi}\cos n\pi x\,dx\right\}$$

$$= 2\left(-\frac{1}{n\pi}\cos n\pi\right) + \frac{4}{n\pi}\int_0^1 (1-x)\cos n\pi x\,dx$$

$$= -\frac{2}{n\pi}\cos n\pi + \frac{4}{n\pi}\left\{\left[(1-x)\frac{1}{n\pi}\sin n\pi x\right]_0^1 - \int_0^1 -1\cdot\frac{1}{n\pi}\sin n\pi x\,dx\right\}$$

$$= -\frac{2}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\left[-\frac{1}{n\pi}\cos n\pi x\right]_0^1$$

$$= -\frac{2}{n\pi}\cos n\pi + \frac{4}{n^3\pi^3}(1 - \cos n\pi)$$